Boltzmann-Gibbs Distribution of Money on Connected Graphs

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Paper

The following presentation is based on a paper titled

Rigorous proof of the Boltzmann-Gibbs distribution of money on connected graphs

by **Nicolas Lanchier**.

Motivation

In equilibrium statistical mechanics it is well-known that the probability p_e that a particle has energy e is well approximated by the exponential random variable (the Boltzmann-Gibbs distribution). That is,

$$p_e \sim \mu e^{-\mu e}, \quad \text{where } \mu = \frac{1}{T} = \text{inverse of temp.}$$

We extend this principle to economics (this is the main idea of econophysics).

- particles = humans
- collisions = interactions

Model

Consider a connected graph $G = (V, E)$ where

- Each vertex represents an economic agent and $N = card(V)$ is the number of agents.
- The edge set E is an interaction network, modeling how the agents interact.
- Each agent is characterized by the amount of money she owns and we let M be the total number of dollars present in the system. This is conserved.

Model

We consider a discrete time Markov chain, ξ_t, that tracks of the amount of money each agent has.

$$\xi_t : V \to \mathbb{N} \quad \text{where } \xi_t(x) \text{ is the number of dollars agent } x \in V \text{ has.}$$

Since total money in the system is conserved, our state space consist of the following subset of spatial configurations:

$$\mathscr{A}_{N,M} = \left\{ \xi \in \mathbb{N} : \sum_{x \in V} \xi(x) = M \right\}$$

Note, we call ξ a particular configuration.

Model

- At each time step choose a directed edge $\{x, y\}$ uniformly at random.
- If $\xi(x) > 0$ (x has at least 1 dollar) we move 1 dollar from vertex (agent) x to y.
- If $\xi(x) = 0$ nothing happens.

That is, our configuration changes in the following way

$$\xi^{\vec{xy}}(z) = \begin{cases} \xi(z) - \chi_{\{z=x\}} + \chi_{\{z=y\}} & \text{when } \xi(x) \neq 0 \\ \xi(z) & \text{when } \xi(x) = 0 \end{cases}$$

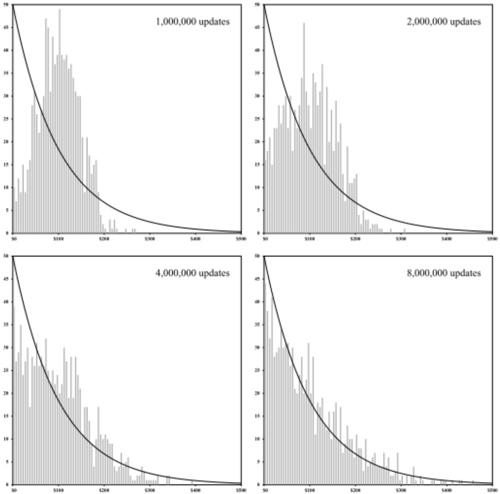
Model Results

Numerical simulations suggest that the probability that an agent has d dollars is well approximated by the exponential random variable (BG distribution). Letting T denote the number of dollars per agent,

$$p_d \sim \mu e^{-\mu d} \quad \text{where } \mu = \frac{1}{T},$$

when the total number of agents and average number of dollars per agent is large.

Model Results



As far as we know, convergence to the exponential distribution has only been shown numerically. **Lanchier provides a rigorous proof.**

Lemma 3

There exists a unique stationary distribution π such that

$$\lim_{t\to\infty} P(\xi_t = \xi) = \pi(\xi), \text{ for any configuration } \xi, \xi_0 \in \mathscr{A}_{N,M}$$

Proof.

If a Markov chain is both irreducible and aperiodic. Then, irreducibility and the fact that the state space $\mathscr{A}_{N,M}$ is finite imply the existence and uniqueness of a stationary distribution π. Furthermore, aperiodicity also implies that, regardless of the initial configuration, the probability that the process is in configuration ξ converges to $\pi(\xi)$. Hence, we need only prove irreducibiliy and aperiodicity. □

Lemma 3 (Irreducibility)

Definition (Irreducibility)

A Markov chain is irreducible if all its states communicate.

Proof

Since G is connected, for any $x, y \in V$ there exists a path $x = x_0, x_1, \cdots, x_n = y$. In particular, for any $\xi \in \mathscr{A}_{N,M-1}$,

$$\begin{aligned}
p_t(\xi^x, \xi^y) &= P(\xi_t = \xi^y \mid \xi_0 = \xi^x) \\
&\geq p(\xi^{x_0}, \xi^{x_1}) p(\xi^{x_1}, \xi^{x_2}) \cdots p(\xi^{x_{n-1}}, \xi^{x_n}) \\
&= p(\xi^{x_0}, (\xi^{x_0})^{\overrightarrow{x_0 x_1}}) p(\xi^{x_1}, (\xi^{x_1})^{\overrightarrow{x_1 x_2}}) \cdots p(\xi^{x_{n-1}}, (\xi^{x_{n-1}})^{\overrightarrow{x_{n-1} x_n}}) \\
&> 0,
\end{aligned}$$

That is, any two configurations ξ^x and ξ^y communicate. By induction, it follows that all configurations communicate because they may be written

$$(\cdots((\xi^{x_1})^{x_2})^{x_3} \cdots)^{x_M} \text{ and } (\cdots((\xi^{y_1})^{y_2})^{y_3} \cdots)^{y_M},$$

Lemma 3 (Irreducibility)

Proof Cont.

where $\xi \in \mathscr{A}_{N,0}$ if the all zero configuration and ξ^{x_0} denotes the configuration obtained by adding a dollar to vertex x_0.

Since all the configurations in $\mathscr{A}_{N,M}$ can be obtained from the all-zero configuration by adding M dollars, we deduce that all the configurations communicate, which by definition means that the process is irreducible. □

Lemma 3 (Aperiodicity)

Definition (Aperiodicity)

An irreducible Markov chain is said to be aperiodic if its period is 1 (how long it takes to return to the same state).

Proof.

Pick $\xi \in \mathscr{A}_{N,M}$ such that $\xi(x) = 0$, then

$$\xi^{\vec{xy}} = \xi \text{ for every } \{x, y\} \in V.$$

That is, for $\xi \in \mathscr{A}_{N,M}$ such that $\xi(z) = 0$ for some $z \in V$,

$$p(\xi, \xi) = \sum_{z \in V} deg(z)\chi_{\{\xi(z)=0\}} \Big/ (2\text{card}(E)) > 0.$$

Thus, configurations with at least one vertex with zero dollar have period one. By irreducibility all the configurations must have the period one, from which it follows that the process is aperiodic. □

Proposition $(*)$

The following proposition is used in the proof of lemma 4.

Proposition $(*)$

If for an irreducible Markov chain with stationary distribution there exists a probability solution π *to*

$$\pi(\xi)p(\xi,\xi') = \pi(\xi')p(\xi',\xi),$$

for all pairs of ξ,ξ'*, then the chain is time-reversible and the solution* π *is the unique stationary distribution.*

Lemma 4

Lemma (4)

The distribution $\pi = U(\mathscr{A}_{N,M})$ is unique and stationary on the state space $\mathscr{A}_{N,M}$

Proof

If $p(\xi, \xi') \neq 0$ then

- $\xi = \xi'$ ('we choose' as vertex with no money)

$$p(\xi, \xi') = p(\xi, \xi) = \sum_{z \in V} deg(z) \chi_{\{\xi(z)=0\}} / (2\text{card}(E)).$$

- $\xi = \eta^x$ and $\xi' = \eta^y$ for some $\eta \in \mathscr{A}_{N,M-1}$ and $\{x, y\} \in E$

$$p(\xi, \xi') = p(\eta^x, \eta^y) = 1/(2\text{card}(E)).$$

Lemma 4

Proof Cont.

This shows that

$$p(\xi,\xi') \neq 0 \iff p(\xi',\xi) \neq 0. \tag{1}$$

When $\pi = U(\mathscr{A}_{N,M})$, $\xi \neq \xi'$, and $p(\xi,\xi') \neq 0$:

$$\pi(\xi)p(\xi,\xi') = \frac{1}{2\text{card}(E)\text{card}(\mathscr{A}_{N,M})} = \pi(\xi')p(\xi',\xi). \tag{2}$$

Note, (2) is trivially true when $\xi = \xi'$ and when true $p(\xi,\xi') = 0$ by (1). This implies that the process is time reversible and that the uniform distribution π is stationary by $(*)$. $\square$

Lemma 5

Lemma (5)

For all positive integers $N, M \in \mathbb{N}$,

$$card(\mathscr{A}_{N,M}) = \binom{M+N-1}{N-1}$$

Proof.

Put $V = \{x_1, \dots, x_N\}$ and for $\xi \in \mathscr{A}_{N,M}$ let

$$\phi(\xi) = \{\xi(x_1)+1, \xi(x_1)+\xi(x_2)+2, \dots, \xi(x_1)+\ldots+\xi(x_{N-1})+N-1\}$$

define a function $\phi : \mathscr{A}_{N,M} \to \mathscr{B}_{N,M}$, where $\mathscr{B}_{N,M}$ is a set of subsets of $\{1, 2, \dots, M+N-1\}$ with $N-1$ elements. If ϕ is bijective it follows that

$$\text{card}(\mathscr{A}_{N,M}) = \text{card}(\mathscr{B}_{N,M}) = \binom{M+N-1}{N-1}.$$

□

Lemma 5

Definition (Recall ϕ)

$\phi(\xi) = \{\xi(x_1) + 1, \xi(x_1) + \xi(x_2) + 2, \ldots, \xi(x_1) + \ldots + \xi(x_{N-1}) + N - 1\}$

Proof.

Injectivity:

Choose $\xi, \xi' \in \mathscr{A}_{N,M}$ such that $\phi(\xi) = \phi(\xi')$. Then, $\xi(x_i) = \xi'(x_i)$ $\forall$ $i = 1, 2, \ldots, N-1$ and $\xi(x_N) = \xi'(x_N)$ as $\xi \in \mathscr{A}_{N,M}$. That is,

$$\phi(\xi) = \phi(\xi') \implies \xi = \xi'.$$

□

Lemma 5

Definition (Recall $\mathscr{B}_{N,M}$)

$\mathscr{B}_{N,M}$ is a set of subsets of $\{1, 2, \ldots, M+N-1\}$.

Proof.

Surjectivity:
Let $B \in \mathscr{B}_{N,M}$ and write $B = \{n_1, n_2, \ldots, n_{N-1}\}$ with $1 \leq n_1 < n_2 < \ldots < n_{N-1} < M+N-1$. Then define the configuration ξ as

$$\xi(x_i) = \begin{cases} n_1 - 1, & \text{for} \quad i = 1 \\ n_i - n_{i-1} & \text{for} \quad i = 2, 3, \ldots, N-1 \\ M + N - n_{N-1} - 1 & \text{for} \quad i = N \end{cases} .$$

One sees that $\xi \in \mathscr{A}_{N,M}$ and $\phi(\xi) = B$, which shows surjectivity. Moreover, ϕ is bijective and this completes the proof of lemma 5. □

Theorem 1 (Exponential Distribution)

Theorem

$$\lim_{t\to\infty} P(\xi_t(x) = d) = \binom{M+N-d-2}{N-2} \Big/ \binom{M+N-1}{N-1}.$$

In particular, for any fixed d,

$$\lim_{N\to\infty} \lim_{t\to\infty} P(\xi_t(x) = d) = \frac{e^{-d/t}}{T} + o\left(\frac{1}{T}\right) \quad \textit{where } T = \frac{M}{N}.$$

Proof of Theorem 1 - part 1

Proof

From Lemmas 3&4 it follows that:

$$\lim_{t\to\infty} P(\xi_t(x) = d) = \pi(\xi \in \mathscr{A}_{N,M} : \xi(x) = d)$$
$$= card\{\xi \in \mathscr{A}_{N,M} : \xi(x) = d\}/card(\mathscr{A}_{N,M})$$

The number of configurations where vertex x has d dollars (the numerator above) is:

$$card\{\xi \in \mathscr{A}_{N,M} : \xi(x) = d\} = card(\mathscr{A}_{N-1,M-d})$$

Proof of Theorem 1 - part 1

Proof cont.

Using Lemma 5 we have that:

$$\lim_{t\to\infty} P(\xi_t(x) = d) = card(\mathscr{A}_{N-1,M-d})/card(\mathscr{A}_{N,M})$$
$$= \binom{M+N-d-2}{N-2} \Big/ \binom{M+N-1}{N-1}$$

This proves the first part of the theorem. □

Proof of Theorem 1 - part 2

Proof cont.

To deduce the second part of the proof, we write the r.h.s. as:

$$\lim_{t\to\infty} P(\xi_t(x) = d) = \frac{M(M-1)\dots(M-d+1)(N-1)}{(M+N-1)(M+N-2)\dots(M+N-d-1)}$$

If we let N go to infinity, we can write:

$$\lim_{N\to\infty}\lim_{t\to\infty} P(\xi_t(x) = d) = \frac{NM^d}{(M+N)^{d+1}}$$

Proof cont.

Let $T = M/N$ be the average number of dollars per vertex which we refer to as the "money temperature":

$$\lim_{N\to\infty}\lim_{t\to\infty} P(\xi_t(x) = d) = \frac{NM^d}{(M+N)^{d+1}} = \left(\frac{1}{T+1}\right)\left(\frac{T}{T+1}\right)^d$$

$$= \left(\frac{1}{T+1}\right) e^{-d\ln\left(1+\frac{1}{T}\right)}$$

Proof of Theorem 1 - part 2

Proof.

For high money temperatures and in the large population limit, we can write:

$$
\begin{aligned}
\lim_{N\to\infty}\lim_{t\to\infty} P(\xi_t(x)=d) &= \left(\frac{1}{T+1}\right) e^{-d\ln\left(1+\frac{1}{T}\right)} \\
&= \left(\frac{1}{T}+o\left(\frac{1}{T}\right)\right) e^{-d\left(\frac{1}{T}+o\left(\frac{1}{T}\right)\right)} \\
&= \frac{e^{-d/T}}{T}+o\left(\frac{1}{T}\right)
\end{aligned}
$$

This shows the exponential random variable $1/T$ approximates the number of dollars for a given vertex, and finishes the proof. □

Recap

Informally, we have shown that for a connected network of agents who randomly exchange a dollar with their neighbours, money distribution is well approximated by the exponential distribution. Moreover, this is true regardless of the network topology.