MTH510: Numerical Analysis **S/S 2021**

Lagrange Form of the Interpolating Polynomial

Lecturer: Saumitra Mazumder

1 Fundamental Curve Fitting Problem

We begin by discussing the fundamental problem of curve fitting.

Given a set of point (x_i, y_i) for $i = 0, 1, 2, \cdots, n$, where each x_i are distinct values of the independent variable and y_i are the corresponding dependent values of some function, say f, we wish to:

1. approximate the value of f at some value x *not* listed among the given x_i

   or

2. determine a function g that in some sense approximates the given data.

The mathematical problem above leads to two different techniques: *interpolation* and *approximation*. In interpolation, we create a function, g, that is determined by enforcing the condition that $g(x_i) = y_i$ for each $i = 0, 1, 2, \cdots, n$.

Interpolation treats error in a *local* manner, whereby the error $\epsilon = |y_i - g(x_i)| = 0$.

Approximation treats error in a *global* manner, whereby the measure of error (say, the sum of the squares of the difference between $g(x_i)$ and y_i) is *minimized*.

For now, we focus solely on the *Lagrange Form of the Interpolating Polynomial*.

Recall a interpolating polynomial that you are familiar with, the Taylor Series:

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \, .$$

The Taylor Series is an interpolating polynomial in the sense that it matches the function and the first n derivative values at the location $x = x_0$. Notice that this polynomial is good for making ***local*** approximations.

2 The Lagrange Form of the Interpolating Polynomial

Let $x_0, x_1, \cdots, x_n$ be $n + 1$ distinct points along the real line. Let $y_0, y_1, \cdots, y_n$ be the function values associated with each x_i.

We seek a polynomial P_n of at most degree n such that $P_n(x_i) = y_i$ for each (x_i, y_i) given.

Example 2.1. *Suppose that we are given a data set consisting of* (x_0, y_0) *and* (x_1, y_1)*, we seek a polynomial* $P_1(x) = a_0 + a_1 x$ *such that*

$$P_1(x_0) = a_0 + a_1(x_0) = y_0 \qquad \textit{and} \qquad P_1(x_1) = a_0 + a_1(x_1) = y_1.$$

Rearranging and substituting, we find the solution of the interpolating conditions are,

$$a_0 = \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} \qquad \textit{and} \qquad a_1 = \frac{y_1 - y_0}{x_1 - x_0}.$$

Hence, the linear interpolating polynomial is

$$P_1(x) = \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x.$$

Rearranging, we have

$$P_1(x) = \frac{x - x_1}{x_1 - x_0} y_0 + \frac{x - x_0}{x_1 - x_0} y_1. \tag{1}$$

We can generalize Eq. (1) to higher degree interpolating polynomials.

Definition 2.2 (The Lagrange Polynomial)**.** *Given data* $\{(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)\}$*, the nth order* ***Lagrange Interpolating Polynomial*** *is*

$$P_n(x) = \sum_{i=0}^{n} L_i(x) y_i, \qquad \textit{where } L_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \tag{2}$$

Example 2.3. *Find the Lagrange interpolation polynomial for* $(1, 0)$*,* $(4, 1.386294)$*,* $(6, 1.791759)$*.*

Solution:

Substituting the given data into Eq. (2), we have

$$\begin{aligned} P_3(x) &= \frac{(x-4)(x-6)}{(1-4)(1-6)}(0) + \frac{(x-1)(x-6)}{(4-1)(4-6)}(1.386294) + \frac{(x-1)(x-4)}{(6-1)(6-4)}(1.791759) \\ &= -0.0518731x^2 + 0.721463x - 0.66959 \end{aligned}$$

Notice that $P_3(1) = 0$, $P_3(4) = 1.386294$ and $P_3(6) = 1.791759$. That is we've fit a polynomial to the data given such that the local error is zero.

We can plot our derived Lagrange Polynomial in MATLAB as follows:

```
>> x = linspace(0,7);y = -0.0518731*x.^2 + 0.721463*x - 0.66959;
>> plot(x,y);
>> hold on
>> plot(1,0, 'O'); plot(4,1.386294, 'O');plot(6,1.791759, 'O'); %data points
```

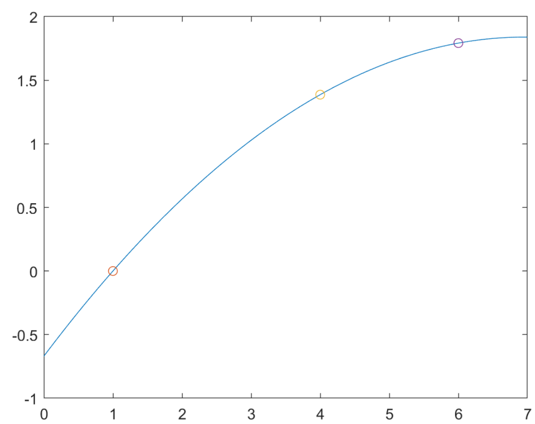

Figure 1: Graph of $P_3(x) = -0.0518731x^2 + 0.721463x - 0.66959$ over the interval $[0, 7]$ and data points given.