Non-Cooperative Games

John Nash

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Outline

Introduction

John Forbes Nash

John Nash generalized the concept of pure strategy equilibrium to mixed-strategy equilibrium. The now eponymous Nash equilibrium is a rational way to define a solution of a non-cooperative game involving two or more players.

We begin by considering some game with n players. In this game, we make the following assumptions:

1. Each player acts independently of each other,
2. There is no collaboration or communication between players,
3. Each player has finitely many turns in the game.

Definitions and Examples

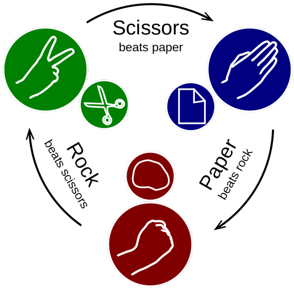
Finite Game

Definition (Finite Game)

(1) An *n-person game* will be a set of *n players* each associated with a finite set of *pure strategies*.

(2) The pure strategy of the *i*th player can be thought about as a plan subject to the observations they make during the course of the game of play; it determines the move a player will make for any situation they could face. Furthermore, a player's strategy set is the set of pure strategies available to that player.

Consider a two person game of rock-paper-scissors, where the winner of the game is the player that wins two out of three individual games: The game is a single move by each player. Each player has the choice of showing either rock, paper or scissors. Each player makes a choice without the knowledge of the other player's choice. Thus, each player has the finite strategy set of $\{rock, paper, scissors\}$.



Rock Paper Scissors

Finite Game Continued

Definition (Finite Game Continued)

(3) Each player, denoted i, has a corresponding *payoff function*, p_i which maps the specific n-tuple of of pure strategies played onto a real number, that is, the n-tuple is always a set of n items with each item associated with a different player.

Finite Game: Example Continued

We can define the following payoff functions:

$$p_i(\pi_1, \pi_2) = 0 \quad \text{whenever } \pi_1 = \pi_2$$

$$p_1(rock, scissors) = 1, \qquad p_2(rock, scissors) = -1$$
$$p_1(scissors, paper) = 1, \qquad p_2(scissors, paper) = -1$$
$$p_1(paper, rock) = 1, \qquad p_2(paper, rock) = -1$$
$$p_1(scissors, rock) = -1, \qquad p_2(scissors, rock) = 1$$
$$p_1(paper, scissors) = -1, \qquad p_2(paper, scissors) = 1$$
$$p_1(rock, paper) = -1, \qquad p_2(paper, rock) = 1$$

Finite Game: Example

Hence, in our game the winner of the individual game is the player that has a payoff of 1 and the losing player has a payoff of -1. Notice that the winner of the game overall is the player that has 1 or more points after three turns, where three points indicates that the player won all three games (the last game is not necessarily played).

Definition: Mixed Strategy, s_i

Definition (Mixed Strategy)

A *mixed strategy* of the ith player is the collection of nonnegative numbers which have a unit sum and have a 1-1 correspondence with the given player's pure strategies.

$$s_i = \sum_{\alpha} c_{i_\alpha} \pi_{i_\alpha} \quad \text{where } c_{i_\alpha} \geqq 0 \text{ and } \sum_{\alpha} c_{i_\alpha} = 1$$

where π_{i_α} represents αth the pure strategy of the ith player.

Definition: Mixed Strategy, s_i

Hence, a mixed strategy can be thought of as an assignment of a probability to each pure strategy.

That is, a mixed strategy is a convex subset of the vector space of pure strategies which are a linear combination of mixed strategies. Furthermore, recall that probabilities are continuous, hence a player has infinitely many mixed strategies.

Definition: Mixed Strategy, s_i

A mixed strategy *uses* a pure strategy π_{i_α} if

$$s_i = \sum_{\beta} c_{i_\beta} \pi_{i_\beta} \quad \text{and } c_{i_\alpha} > 0$$

If $\mathbb{S}$ uses pure strategy π_{i_α}, we can also say that $\mathbb{S}$ uses π_{i_α}

Mixed Strategy: A Less Trivial Example

Recall the notion of a breakaway in hockey. A nongoalie player is alone in competition with the goalie.

Suppose the nongoalie player has only two choices, to shoot to the right or left side of the goal. Simultaneously, the goalie must choose whether to which way to move to block the shot, again right or left. Assume that if the goalie guesses correctly, the shot is always blocked and the payoff for each player is 0.

Breakaway

Now, suppose further that the nongoalie player plays with a left handed stick (ie, left hand at the bottom of the stick) and thus is a better shooter going forehand (ie, shooting right) vs. backhand (ie, shooting left).

If the goalie guesses wrong, the shot from the nongoalie player is more likely to go into the net when shot rightward than leftward.

Mixed Strategy: A Less Trivial Example

Hence, we have the strategy set $\{left, right\}$ for both players

$$p_i(\pi_{1\alpha}, \pi_{2\beta}) = 0 \quad \text{whenever } \pi_{1\alpha} = \pi_{2\beta}$$
$$p_1(right, left) = 2, \quad p_2(right, left) = -2$$
$$p_1(left, right) = 1, \quad p_2(left, right) = -1$$

		Goalie	
		Left	Right
Nongoalie	Left	0, 0	1, −1
	Right	2, −2	0, 0

Table: Payoff Matrix

Mixed Strategy: A Less Trivial Example

Now, consider the perspective of the goalie. The goalie knows that the left handed nongoalie will choose to shoot rightward more often, since the nongoalie has a higher payoff when shooting rightward.

Suppose the goalie thinks that the nongoalie will shoot rightward with a probability k. The goalie's expected payoff for blocking right is $k \times 0 + (1 - k) \times (-1)$ and the expected payoff for blocking left is $(1 - k)0 + k \times (-2)$.

Equilibrium Point

Denote the n-tuple of mixed strategies of each player as $\mathbb{S}$. Then the payoff of this ith player, is

$$p_i(\mathbb{S}) := p_i(s_1, s_2, \cdots s_n).$$

For convenience, if we want to focus on the mixed strategy of the ith player, we consider $(\mathbb{S}; t_i) := (s_1, s_2, \cdots, s_{i-1}, t_i, s_{i+1}, \cdots, s_n)$.

We can of course consider multiple players via successive substations as follows: $((\mathbb{S}; t_i); r_j) = (\mathbb{S}; t_i; r_j)$

Equilibrium Point

Theorem

An n-tuple $\mathbb{S}$ is an equilibrium point iff for every i

$$p_i(\mathbb{S}) = \max_{all\ r_i} \{p_i(\mathbb{S}; r_i)\} \tag{1}$$

Equilibrium Point: A Less Trivial Example

Returning to our hockey breakaway example:
Notice that this game has no pure-strategy equilibrium, since, say the nongoalie, would always deviate from a pure strategy.

		Goalie	
		Left	Right
Nongoalie	Left	0, 0	1, -1
	Right	2, -2	0,0

Table: Payoff Matrix

Equilibrium Point: A Less Trivial Example

For example, $(Left, Left)$ is not an equilibrium because the nongoalie would deviate to Right and increase their payoff from 0 to 1.

		Goalie	
		Left	Right
Nongoalie	Left	0, 0	1, -1
	Right	2, -2	0,0

Table: Payoff Matrix

Equilibrium Point

Supposes that $\mathbb{S}$ uses π_{i_α}. Define $p_{i_\alpha}(\mathbb{S}) := p_i(\mathbb{S}; \pi_{i_\alpha})$. From the linearity of $p_i(s_1, s_2, \cdots, s_n)$ in s_i, we have

$$\max_{\text{all } r_i} \{p_i(\mathbb{S}; r_i)\} = \max_{\alpha} \{p_i(\mathbb{S}; \pi_{i_\alpha})\} \tag{2}$$

Equilibrium Point

This allows us to obtain the necessary and sufficient condition

$$p_i(\mathbb{S}) = \max_{\alpha} p_{i_\alpha}(\mathbb{S}) \tag{3}$$

Definition: Equilibrium Point

For $s_i = \sum_\alpha c_{i_\alpha}\pi_{i_\alpha}$, notice that $p_i = \sum_\alpha c_{i_\alpha} p_{i_\alpha}(\mathbb{S})$. Furthermore, for equation (3) to hold, we must have $c_{i_\alpha} = 0$ whenever $p_{i_\alpha}(\mathbb{S}) < \max_\beta p_{i_\beta}(\mathbb{S})$. That is $\mathbb{S}$ does not use pure strategy π_{i_α} unless it is the optimal pure strategy for the ith player.

Hence, another necessary and sufficient condition for an equilibrium point is

$$\text{if } \pi_{i_\alpha} \text{ is used in } \mathbb{S} \text{ then } p_{i_\alpha}(\mathbb{S}) = \max_\beta p_{i_\beta}(\mathbb{S}) \tag{4}$$

Results

Theorems

There are five theorems that are proved in this paper. Three of them concern equilibrium points and the remaining two concern the solvability of the game.

Theorem 1

Theorem

(Theorem 1) Every finite game has an equilibrium point.

Theorem 1

Proof.

The proof proceeds by defining the following:

1. $\mathbb{S}$ – An $n-$tuple of mixed strategies.
2. $p_i(\mathbb{S})$ – The corresponding pay-off to player i.
3. $p_{i_\alpha}(\mathbb{S})$ – the pay-off to player i if they change to the αth pure strategy π_{i_α} with the other players holding to their respective mixed strategies.
4. $\varphi_{i_\alpha} = \max(0, p_{i_\alpha}(\mathbb{S}) - p_i(\mathbb{S}))$ – A set of continuous functions of $\mathbb{S}$.
5. $s'_i = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\mathbb{S})\pi_{i_\alpha}}{1 + \sum_\alpha \varphi_{i_\alpha}(\mathbb{S})}$
6. $\mathbb{S}$' – The n-tuple $(s'_1, s'_2, \ldots, s'_n)$

□

Proof of Theorem 1 cont...

Proof.

The proof proceeds by showing that the fixed points of the mapping $T : \mathbb{S} \to \mathbb{S}'$ are the game's equilibrium points.

It utilizes the idea that a mixed strategy is essentially a linear combination of pure strategies and that there is a least profitable pure strategy.

If $\mathbb{S}$ is fixed under T, no player can improve their pay-off by moving to another pure strategy π_{i_β}, which is a criterion for an equilibrium point.

The key insight is that $\mathbb{S}$ being fixed under T implies that $\mathbb{S}$ is a fixed point, which in turn implies that $\mathbb{S}$ is an equilibrium point. □

Theorem 2

Theorem

(Theorem 2) Any finite game has a symmetric equilibrium point.

A *symmetry* is a permutation of pure strategies. The permutation of pure strategies induces a permutation of the players. Essentially what this implies is that each player is using the same strategy.

Theorem

(Theorem 3) A sub-solution $\mathfrak{S}$ *is the set of all n-tuples* $(s_1, s_2, \ldots, s_n)$ *such that each* $s_i \in S_i$ *where* S_i *is the* i^{th} *factor set of* $\mathfrak{S}$*. Geometrically,* $\mathfrak{S}$ *is the product of its factor sets.*

A game is said to be *solvable* if its set $\mathfrak{S}$ of equilibrium points is such that

$$(\mathfrak{t}; r_i) \in \mathfrak{S} \text{ and } s \in \mathfrak{S} \rightarrow (s; r_i) \in \mathfrak{S}$$

Essentially this says that there is an n-tuple where each strategy is maximal relative to the set of equilibrium points.

Theorem 4

Theorem

(Theorem 4) The factor sets $S_1, S_2, \ldots, S_n$ of a sub-solution are closed and convex as subsets of the mixed strategy spaces.

Theorem 5

Theorem

(Theorem 5) The sets $S_1, S_2, \ldots, S_n$ *of equilibrium strategies in a solvable game are polyhedral convex subsets of the respective mixed strategy spaces.*

Thank You

Questions?

Backup slides go here