AM8209: Functional Analysis **S/S 2021**

Functional Analysis

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Contents

1 Linear Spaces: Definition, Examples and Linear Span

1.1 Linear Spaces

Definition 1.1 (Field). *A **field** is a set $\mathbb{K}$, containing at least two elements, on which the addition and multiplication are defined as usual, such that for each pair of elements x, y in $\mathbb{K}$ the elements $x + y$ and xy are also in $\mathbb{K}$.*

Aside: In general, when we speak of sets, we refer to *nonempty* sets unless otherwise specified.

Definition 1.2 (Linear Space). *X is **linear space** over a field $\mathbb{K}$, if X is a set equipped with two operations,* addition *and* scalar multiplication *defined as follows:*

If x and y are elements of a linear space X, then $x + y$ is also an element in X. The operation $+$ has the following properties:

$+$ is commutative:

$$x + y = y + x \tag{1}$$

$+$ is associative: for elements x, y, z in X,

$$x + (y + z) = (x + y) + z \tag{2}$$

X is a group. That is, there exists an additive identity element, denoted 0, in X such that

$$0 + x = x \tag{3}$$

and there is an additive inverse element of x, denoted as $-x$, in X such that

$$-x + x = 0 \tag{4}$$

For an element x belonging to a linear space X, $\alpha \cdot x$ also belongs to X. The operation $\cdot$ has the following properties:

$\cdot$ is associative:

$$(\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x) \tag{5}$$

$\cdot$ is distributive:

$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x \tag{6}$$

and

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y \tag{7}$$

The neutral element represented by 1 is the element of the field $\mathbb{K}$ such that

$$1x = x \tag{8}$$

For convenience, we suppress $\cdot$ and simply write αx.

The finite-dimensional linear spaces are dealt with in courses on linear algebra. These lectures emphasize linear spaces that are *not* finite-dimensional.

Unless stated otherwise, the field $\mathbb{K}$ will be either real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

Property 1.3 (Properties of Linear Spaces)**.**

1. *For* 0 *in* $\mathbb{K}$ *and* x *in* X, $0x = 0$ *is also in* X.
   Proof:

$$\begin{aligned} &0x = (0+0)x = 0x + 0x \ Eq. \ (6) \\ &-0x + 0x = -0x + 0x + 0x \ Eq. \ (4) \\ &0 = 0x. \end{aligned}$$

■

2. *For* x *and* $-x$ *in* X, $-1x = -x$.
   Proof:

$$\begin{aligned} 0 = 0x = (-1+1)x = -1x + 1x &= -1x + x \ Eq. \ (6) \\ 0 + -x &= -1x + x + -x \ Eqs. \ (3) \ and \ (4) \\ -x &= -1x. \end{aligned}$$

■

Example 1.4 (Examples of Linear Spaces)**.**

1. $X = \{p(t) : polynomial\ t \in \mathbb{R}\}$ *and* $\mathbb{K} = \mathbb{R}$.

   Solution. *Notice that the sum and scalar multiple of polynomials are still polynomials. That is, the space* $\{p(t) : polynomial\ t \in \mathbb{R}\}$ *and* $\mathbb{K} = \mathbb{R}$ *is a linear space.*

2. $\mathbb{R}^N$ *and* $X = C(\mathbb{R}^N)$ *as the space of continuous functions and* $\mathbb{K}$ *is either the complex plane or the set of real numbers.*

   Solution. *Fix some interval* $[a, b]$ *such that* $-\infty < a < b < \infty$. *Pick two functions, say* $u : [a, b] \mapsto \mathbb{R}$ *and* $v : [a, b] \mapsto \mathbb{R}$.

   Now, for $u, v \in C[a, b]$ *and* $\alpha \in \mathbb{R}$, *put* $(u+v)(x) = u(x)+v(x)$ *and* $(\alpha u)(x) = \alpha u(x)$ *for any* $x \in [a, b]$.

   Notice that the sum and scalar product of continuous functions on the same domain are again continuous. That is, $(u + v)(x)$ *and* $(\alpha u)(x)$ *are also in* $C[a, b]$.

   Hence, $C[a, b]$ *is a linear space.*

3. $X = \{(a_j)_{j\geq 1} = (a_1, a_2, \cdots) : a_j \in \mathbb{K}\}$ *and* $\mathbb{K}$ *is either the complex plane or the set of real numbers.*

   Solution. *Notice that the sum and scalar product of elements of the complex plane are also in the complex plane. Hence,* $\{(a_j)_{j\geq 1} = (a_1, a_2, \cdots) : a_j \in \mathbb{K}\}$ *is a linear space.*

4. *A sigma field* $(\Omega, \mathcal{F}, \mu)$ *and* $X = L^p = \{f : \Omega \mapsto \mathbb{R} : \int_\Omega |f|^p \, d\mu < \infty\}$ *and* $\mathbb{K} = \mathbb{R}$.

   Fix p *and pick functions* f *and* g *from* $\{f : \Omega \mapsto \mathbb{R} : \int_\Omega |f|^p \, d\mu < \infty\}$. *Notice that* $|f + g|^p \leq |f|^p + |g|^p$ *and* $\alpha |f|^p \leq |\alpha f|^p$. *That is,* $\int_\Omega |f + g|^p \, d\mu \leq \int_\Omega |f|^p \, d\mu + \int_\Omega |g|^p \, d\mu < \infty$ *and* $\alpha \int_\Omega |f|^p \, d\mu \leq \int_\Omega |\alpha f|^p \, d\mu < \infty$ *for any* $\alpha \in \mathbb{K}$.

Example 1.5 (Examples of linear spaces from [1])**.**

(i) *X is the space of all polynomials in a single variable s, with real coefficients, here* $\mathbb{K} = \mathbb{R}$.

(ii) *X is the space of all polynomials in N variables $S_1, \cdots, s_N$, with real coefficients, here $\mathbb{K} = \mathbb{R}$.*

(iii) *G is a domain in the complex plane, and X the space of all functions complex analytic in G, here $\mathbb{K} = \mathbb{C}$*

(iv) *X = space of all vectors, $x = (a_1, a_2, \cdots)$ with infinitely many real components, here $\mathbb{K} = \mathbb{R}$.*

(v) *Q is a Hausdorff space, X the space of all continuous real-valued functions on Q, here $\mathbb{K} = \mathbb{R}$.*

(vi) *M is a C^∞ differentiable manifold, $X = C^\infty(M)$, the space of differentiable functions on M.*

(vii) *Q is a measure space with measure m, $X = L^1(Q, m)$.*

(viii) *$X = L^P(Q, m)$.*

(ix) *X = harmonic functions in the upper half-plane.*

(x) *X = all solutions of a linear partial differential equation in a given domain.*

(xi) *All meromorphic functions on a given Riemann surface; $\mathbb{K} = \mathbb{C}$.*

1.1.1 Linear Subspaces

Definition 1.6 (Linear Subspace)**.** *For a linear space X over a field $\mathbb{K}$, $Y \subseteq X$ is a* ***linear subspace*** *(also called a* ***sub-linear space****) if x and y are in Y and α in $\mathbb{K}$, then $\alpha x + y$ is also in Y.*

Remark 1.7. *The additive identity element, 0, of X is an element a subspace Y of linear space X and is also the additive identity element of Y.*

Proof: Consider an element y in Y. Put $\alpha = -2$. Then $-2 \cdot y + y = -y$ is an element of Y. Put $\alpha = -1$, then $-1 \cdot y + y = 0$ is an element of Y. Finally, put $\alpha = 1$, then $1 \cdot y + 0 = y$ is an element of Y.

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Definition 1.8. *If A and B are subsets of linear subspace Y of linear space X, the for α and β in $\mathbb{K}$, the set $\alpha A + \beta B$ is defined as $\alpha A + \beta B = \{z \in X : z = \alpha x + \beta y, \text{ for } x \in A, y \in B\}$.*

Property 1.9 (Properties of Linear Subspaces)**.**

1. *$\{0\}$ and X are linear subspaces of X.*

   Solution.

   (a) *Notice that $0 = \alpha 0 + 0$, for any $\alpha \in K$. Hence, $\{0\}$ is a sublinear space.*

   (b) *It is clear that $X \subseteq X$. Since X is a linear space, we have that X is also a sublinear space of X.*

2. *The sum of any collection of subspaces is a subspace.*

   Proof: *Notice, if Y_1 and Y_2 are linear subspaces, then $Y_1 + Y_2 = \{x + y : x \in Y_1 \text{ and } y \in Y_2\}$ is also a linear subspace. Consider two arbitrary elements x and y in $Y_1 + Y_2$ and α in $\mathbb{K}$. Notice by the definition of $Y_1 + Y_2$, we have $x = x_1 + x_2$ and $y = y_1 + y_2$ where x_1 and y_1 are in Y_1 and x_2 and y_2 is in Y_2. Furthermore, notice, $\alpha x_1 + y_1 \in Y_1$ and $\alpha x_2 + y_2 \in Y_2$, since Y_1 and Y_2 are linear subspaces.*

   Then,

$$\begin{aligned}(\alpha x_1 + y_1) + (\alpha x_2 + y_2) &= \alpha x_1 + \alpha x_2 + y_1 + y_2 \\ &= \alpha(x_1 + x_2) + (y_1 + y_2) \\ &= \alpha x + y \in Y_1 + Y_2.\end{aligned}$$

By induction we can show that $\sum_{i \in I} Y_i$ of any collection of linear subspaces, $\{Y_i : i \in I\}$ is also a linear subspace.

■

3. *Assume there is a family of linear subspaces of X denoted $\{Y_\theta : \theta \in I\}$. Then $Y = \bigcap_{\theta \in I} Y_\theta$ is also a linear subspace.*

   Proof: *Fix x and y in $Y = \bigcap_{\theta \in I} Y_\theta$, then x and y are in a fixed Y_θ. Since Y_θ is a sub-linear space, we have that $\alpha x + y$ in is Y_θ. Since θ was arbitrary, it holds that $\alpha x + y$ in is* every *Y_θ for $\theta \in I$. Hence $\alpha x + y \in \bigcap_{\theta \in I} Y_\theta$.*

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Definition 1.10 (Totally Ordered)**.** *$\{Y_\theta : \theta \in I\}$ is* ***totally ordered*** *if θ_1 and θ_2 in I then $Y_{\theta_1} \subseteq Y_{\theta_2}$ or $Y_{\theta_2} \subseteq Y_{\theta_1}$.*

4. For a *totally ordered* family of sub-linear spaces $\{Y_\theta : \theta \in I\}$, we have that $\bigcup_{\theta \in I} Y_\theta$ is also a linear sub-space.

   Proof: Consider x and y in the *totally ordered* family of sub-linear spaces $\bigcup_{\theta \in I} Y_\theta$, then there exists θ_1 and θ_2 such that $x \in Y_{\theta_1}$ and $y \in Y_{\theta_2}$. Since the family of sub-linear spaces are totally ordered, we have either $Y_{\theta_1} \subseteq Y_{\theta_2}$ or $Y_{\theta_2} \subseteq Y_{\theta_1}$. Without loss of generality, assume that $x \in Y_{\theta_1} \subseteq Y_{\theta_2}$ So both x and y are in Y_{θ_2}. Since Y_{θ_2} is a sub-linear space, we have that $\alpha x + y$ in is Y_{θ_2}. Hence $\alpha x + y \in \bigcup_{\theta \in I} Y_\theta$.

■

1.2 Linear Spans

Definition 1.11 (Linear Span)**.** *For a family of linear subspaces of X, $\{Y_\theta : \theta \in I\}$, over a field $\mathbb{K}$, consider a set $S \subseteq Y_\theta$ for each $\theta \in I$. The* ***Linear Span*** *of S, denoted $LS(S)$, is $LS(S) = \bigcap_{\theta \in I} Y_\theta$.*

Property 1.12 (Properties of Linear Spans)**.** *$LS(S)$ is the* smallest linear space *that contains S.*

Proof: Recall from Property 1.9 that $LS(S)$ is a linear subspace.

Now, suppose $\{Y_\theta : \theta \in I\}$ is a collection of linear subspaces of X such that $S \subseteq Y_\theta$ for each $\theta \in I$. Notice that $Y_\theta \supseteq LS(S) = \bigcap_{\theta \in I} Y_\theta$. Since Y_θ was arbitrary, it follows that *any* linear subspace that contains S must also contain $LS(S)$. That is, $LS(S)$ is the *smallest linear space* that contains S.

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Theorem 1.13 (Linear Span)**.** *For a linear space X over a field $\mathbb{K}$ and a set $S \subseteq X$,* ***linear span of*** *S, $LS(S)$, consists of all possible linear sums of elements of S. That is*

$$LS(S) = \left\{ \sum_{j=1}^{n} \alpha_j x_j : \alpha_j \in \mathbb{K}, x_j \in S, n \text{ is any natural number} \right\} \tag{9}$$

Proof: Put $Z = \left\{ \sum_{j=1}^{n} \alpha_j x_j : \alpha_j \in \mathbb{K}, x_j \in S, n \text{ is any natural number} \right\}$. We want to show that $LS(S) = Z$.

In particular, we want to show:

1. Z contains S, and

2. Z is a linear subspace.

If Z is a linear subspace that containing S then by Property 1.12, we will have $Z \supseteq LS(S)$.

Finally, we will show

3. If Y is an arbitrary linear subspace that contains S, then $Y \supseteq Z$.

Now,

1. Z contains S is clear, since if we take any element of S, say x. Then $x = 1x$ is of the form Eq. (9). Hence Z contains S.
2. Pick z_1, z_2, in Z and α in K. Then $z_1 = \sum_{j=1}^{N_1} \alpha_j x_j$ for $\alpha_j \in \mathbb{K}, x_j \in S$, and $z_2 = \sum_{j=1}^{N_2} \beta_j y_j$ for $\beta_j \in \mathbb{K}, y_j \in S$.
   Now, without loss of generality for $N_2 \geq N_1$, we have
   $\alpha z_1 + z_2 = \sum_{j=1}^{N_1} \alpha\alpha_j x_j + \sum_{j=1}^{N_2} \beta_j y_j = \sum_{j=1}^{N_1} (\alpha\alpha_j x_j + \beta_j) + \sum_{j=1}^{N_2 - N_1} \beta_j y_j$.
   Notice that each $\alpha\alpha_j$ belongs to $\mathbb{K}$. Since $\sum_{j=1}^{N_1} (\alpha\alpha_j x_j + \beta_j) + \sum_{j=1}^{N_2 - N_1} \beta_j y_j$ is of the form Eq. (9), we have that $\alpha z_1 + z_2 \in Z$. Hence, Z is a linear space.

Therefore by Property 1.12, we have $Z \supseteq LS(S)$

3. Consider an arbitrary linear subspace $Y \supseteq S$ and pick $z \in Z$. By definition of Z, $z = \sum_{j=1}^{n} \alpha_j x_j$ for some $n \in \mathbb{N}$. Since Y is a linear subspace which contains S, we have each $\alpha_j x_j \in Y$. Furthermore, $\sum_{j=1}^{N} \alpha_j x_j$ is an element of Y, since it is a linear combination of elements of S. Hence $Y \supseteq Z$.

So, we have shown that for any linear subspace Y which contains S, we have that $Y \supseteq Z$. But by Property 1.12, $LS(S)$ is the *smallest* linear subspace containing S. Hence, $Z = LS(S)$.

■

Remark 1.14 (Remark 1 from [1]). *An element x of the form in Eq.* (9) *is called a* linear combination *of the points $x_1, \cdots, x_n$ of elements of S. So Theorem 1.13 can be restated as follows:*

The linear span of a subset S of a linear space X consists of all linear combinations of elements of S [1].

1.3 Problems from Section 1

Problem 1.15. *Prove the first part of Property 1.9.*

Problem 1.16. *Show that each Example 1.4 are linear spaces.*

Problem 1.17. *Show that each Example 1.5 are linear spaces.*

2 Linear Spaces: Quotient Spaces and Convex Sets

2.1 Quotient Spaces

Definition 2.1 (Equivalent modulo). *For a linear subspace Y of linear space X over a field $\mathbb{K}$, we introduce a relation, $(\sim)$, between two elements of X . Two arbitrary elements in X, say x, y, are **equivalent modulo**,* *denoted $x \sim y$, if $x - y$ is also in Y.*

Claim 2.2. *The relation $(\sim)$ is an equivalence relation.*

Proof: Suppose Y is a linear subspace of a linear space X.

$$x \sim x \quad \text{(Reflexive)}. \tag{10}$$

Pick an arbitrary element of X, say x. Since X is a linear space, we have $-x$ is also in X. Then, we have $-x + x = x - x = 0$ is in Y by definition of Y. Hence $x \sim x$.

$$\text{If } x \sim y, \text{then } y \sim x \quad \text{(Symmetric)}. \tag{11}$$

Suppose $(x - y)$ is an element of Y. Then have $-1(x - y) = -x + y = (y - x)$ is also an element of Y, since Y is a linear subspace. Hence, if $x \sim y$, then $y \sim x$.

$$\text{If } x \sim y \text{ and } y \sim z, \text{then } x \sim z \quad \text{(Transitive)}. \tag{12}$$

Suppose $(x-y)$ and $(y-z)$ are elements of Y . Since Y is a linear subspace. We have $(x-y)+(y-z) = (x-z)$ is also an element of Y. Hence, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Hence, by Eqs. (10) to (12) $\sim$ is an equivalence relation.

■

We can divide the linear space X into distinct equivalence classes modulo linear subspace Y. We denote this set of equivalence classes as $X \mid_Y$ or X mod Y.

Definition 2.3 (Quotient Space)**.** *For x, y in a linear space X, and a sublinear space Y, we define the following:*

1. *$[x] = \{y \in X : x \sim y\} = \{y \in X : (x - y) \in Y\}$. That is for $x \in X$, $[x]$ represents all the elements of $y \in X$ such that the linear combination $(x - y) \in Y$.*
2. *We represent the* ***Quotient Space*** *of X modulo Y as $X \mid_Y = \{[x] : x \in X\}$. That is $X \mid_Y$ is the set of all equivalence classes of elements of X.*

Remark 2.4. *From Definitions 2.1 and 2.3 it is clear that any representative from an equivalence class can be used to denote that equivalence class. That is, for any element $z \in [x]$, $[z] = [x]$.*

Proof: Pick $z \in [x]$. Then $(z - x) \in Y$. Since Y is a linear subspace, $-1(z - x) = x - z$ is also an element of Y. That is $x \in [z]$. Hence, $[x] \subseteq [z]$. By symmetry, $[x] \supseteq [z]$. Thus, $[z] = [x]$, for any $z \in [x]$ ■

For $X \mid_Y$ to be a linear space, we need to define the sums as well as scalar multiplication of elements of $X \mid_Y$, denoted $[x] + [y]$ and $\alpha[x]$ respectively, for elements $[x], [y] \in X \mid_Y$ and $\alpha \in \mathbb{K}$ such that both $[x] + [y]$ and $\alpha[x]$ are in $X \mid_Y$.

Claim 2.5 (+)**.** *Notice that both $[x]$ and $[y]$ are* subsets *of the linear space X. Put $[x] + [y] = [x + y]$. Then the operation $[x] + [y] = [x + y]$ is* well-defined.

Proof: To show (+) is well-defined, we will show that for x_1, x_2 in $[x]$ and y_1, y_2 in $[y]$, we have $[x_1 + y_1] = [x_2 + y_2]$.

1. Suppose $x_1, x_2 \in [x]$ and $y_2, y_2 \in [y]$. Then, from the definition of $[x]$ and $[y]$, we have $(x_1 - x), (x_2 - x), (y_1 - y), (y_2 - y)$ are elements of Y. Hence, by Claim 2.2, we have that $(x_2 - x_1), (y_2 - y_1)$ are also elements of Y.

2. Now, pick an arbitrary z from $[x_1 + y_1]$, we want to show that z is also in $[x_2 + y_2]$.

   Notice that $z - (x_1 + y_1)$ is an element of Y by the definition of $[x_1 + y_1]$. But Y is a linear space, so we have

$$\begin{aligned} z - (x_1 + y_1) &= z - (x_1 + x_2 - x_2 + y_2 - y_2 + y_1) \\ &= z - (x_2 + y_2) + (x_2 - x_1) + (y_2 - y_1) \quad \text{is an element of } Y. \end{aligned} \tag{13}$$

   Furthermore, since $(x_2 - x_1), (y_2 - y_1)$ are elements of the linear subspace Y from (1), we have

$$\begin{aligned} \Big(z - (x_1 + y_1)\Big) - (x_2 - x_1) - (y_2 - y_1) &= z - \Big((x_2 + y_2) + (x_2 - x_1) + (y_2 - y_1)\Big) - (x_2 - x_1) - (y_2 - y_1) \\ &= z - (x_2 + y_2) \quad \text{is an element of } Y. \end{aligned}$$

   In particular, $z \sim (x_2 + y_2)$. Thus, $[x_1 + y_1] \subseteq [x_2 + y_2]$. Furthermore, by symmetry, the converse is also true. That is, $[x_1 + y_1] = [x_2 + y_2]$.

Hence, $[x + y] = [x + y]$ is well-defined. ■

Claim 2.6 $(\cdot)$**.** *For an arbitrary element from X, say x, and $\alpha \in \mathbb{K}$, put $\alpha[x] = [\alpha x]$. The operation $\alpha[x] = [\alpha x]$ is* well-defined.

Proof: To show $(\cdot)$ is well-defined, we will show that for x_1, x_2 in $[x]$ and $\alpha \in \mathbb{K}$, we have $[\alpha x_1] = [\alpha x_2]$.

1. Suppose $x_1, x_2 \in [x]$. Then $(x_1 - x), (x_2 - x) \in Y$. Since, Y is a linear subspace, $(x_1 - x) - (x_2 - x) = (x_1 - x_2)$ is also in Y.

2. Pick an arbitrary element z from $[\alpha x_1]$. Then $z - \alpha x_1 \in Y$. Since Y is a linear subspace, $z - \alpha x_1 - \alpha(x_1 - x_2) = z - \alpha x_2$ is also in Y. That is $z \in [\alpha x_2]$. Hence, $[\alpha x_1] \subset [\alpha x_2]$. By symmetry, $[\alpha x_1] \supset [\alpha x_2]$. That is, $[\alpha x_1] = [\alpha x_2]$.

Hence, $\alpha[x] = [\alpha x]$ is well defined.

■

2.2 Linear Maps

As with all algebraic structures, linear structures we have the concept of *isomorphism.*

Definition 2.7 (Linear Maps)**.** *For a linear spaces X^*, X over a field $\mathbb{K}$, the mapping $T : X \mapsto X^*$ is a **linear map** if:*

1. *$T(x + y) = T(x) + T(y)$, and*
2. *$\alpha T(x) = T(\alpha x)$, for $\alpha \in \mathbb{K}$*

Definition 2.8 (Isomorphism)**.** *X and X^* are **isomorphic** if there exists a linear map $T : X \mapsto X^*$ which is a bijection.*

For a fixed linear map $T : X \mapsto X^*$, we have the following claims:

Claim 2.9. *For a linear subspace Y of linear space X over a field $\mathbb{K}$, $T(Y) = \{T(x) : x \in Y\}$ is a linear subspace.*

Proof: To show that $T(Y)$ is a linear map, we want to show that for $\alpha \in \mathbb{K}$ and elements z_1 and z_2 in $T(Y)$, we have that $\alpha z_1 + z_2$ is also an element of $T(Y)$.

Notice, if α in $\mathbb{K}$ and z_1 and z_2 in $T(Y)$, there are elements x_1 and x_2 in Y such that $T(x_1) = z_1$ and $T(x_2) = z_2$. Since Y is a linear subspace, we have $\alpha x_1 + x_2$ is in Y. Hence $T(\alpha x_1 + x_2)$ in $T(Y)$. But T is a linear map, so $T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2) = \alpha z_1 + z_2$.

Hence, $\alpha T(x_1) + T(x_2) = \alpha z_1 + z_2$ is an element of $T(Y)$

■

Claim 2.10. *For a linear subspace Y^* of linear space X^* and another linear space X both over a field $\mathbb{K}$. Denote $T^{-1}(Y^*)$ as the inverse image of the previously fixed linear map T.*

That is, $T^{-1}(Y^*) = \{x \in X : T(x) \in Y^*\}$. Then, $T^{-1}(Y^*)$ is a linear subspace.

Proof: Consider two arbitrary elements, z_1 and z_2 in $T^{-1}(Y^*)$.

Notice, by definition of T^{-1}, we have that $T(z_1)$ and $T(z_2)$ are elements of Y^*. Then, by linearity of T, we have $T(\alpha z_1 + z_2) = \alpha T(z_1) + T(z_2)$ is an element of Y^*, since Y^* is a linear subspace.

In particular $\alpha z_1 + z_2$ is in $T^{-1}(Y^*)$.

■

2.3 Convex Sets

A very important concept in a linear space over the reals is *convexity.*

Definition 2.11 (Convex Sets)**.** *For a linear space X over a field $\mathbb{K} = \mathbb{R}$, a subset $K \subseteq X$ is said to be* ***convex*** *if for any two elements x and y in K and $\alpha \in [0, 1]$, we have $\alpha x + (1 - \alpha)y$ is in K.*

Example 2.12. *Examples of convex sets in the plane are (1) circular disk, (2) triangle, and (3) semicircular disk.*

Solution (Example 2.12)**.**

1. *Without loss of generality, fix a circle with radius r centred at $(0, 0)$ on the $\mathbb{R}^2$-plane. Then for any element in this circle is of the form $|x| \leq r$. Now, Pick two arbitrary ordered pairs from this circle, say $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. A convex combination of those elements is $\alpha P_1 + (1 - \alpha)P_2 = \Big(\alpha x_1 + [1 - \alpha]x_2, \alpha y_1 + [1 - \alpha]y_2\Big)$. Since $|x| \leq r$ we have $\Big|\alpha x_1 + [1 - \alpha]x_2\Big| \leq r$. That is the convex combination of any two elements in a circle is also an element of the same circle.*

Claim 2.13 (Convex Combination)**.** *If K is convex, $x_1, \cdots, x_N$ in K and $\alpha_1, \cdots, \alpha_N$ in $[0, 1]$ such that $\sum_{j=1}^{N} \alpha_j = 1$, then $\sum_{j=1}^{N} \alpha_j x_j$ is also in K.*

The above is called a ***convex combination*** *of elements of K.*

Proof: Notice that $x = 1x + 0x$ and $\alpha x + (1 - \alpha)y$ are convex by Definition 2.11 when $\alpha \in [0, 1]$ for arbitrary elements x, y in convex set K.

Suppose that linear combinations of the form $\sum_{j=1}^{N} \alpha_j x_j$ are in the convex set K.

A convex combination of arbitrary elements $x_1, \cdots, x_{N+1}$ of K is $\sum_{j=1}^{N+1} \beta_j x_j$, for $\beta \in [0,1]$.

Now,

$$\sum_{j=1}^{N+1} \beta_j x_j = \sum_{j=1}^{N} \beta_j x_j + \beta_{N+1} x_{N+1}.$$

Notice that $\beta_{N+1} = 1 - \sum_{j=1}^{N} \beta_j$.

Then we have,

$$\begin{aligned}\sum_{j=1}^{N+1} \beta_j x_j &= \sum_{j=1}^{N} \beta_j x_j + \left(1 - \sum_{j=1}^{N} \beta_j\right) x_{N+1} \\ &= \sum_{j=1}^{N} \beta_j \left(\frac{1}{\sum_{j=1}^{N} \beta_j} \sum_{j=1}^{N} \beta_j x_j\right) + \left(1 - \sum_{j=1}^{N} \beta_j\right) x_{N+1}\end{aligned}$$

Since $\sum_{j=1}^{N} \beta_j x_j$ is an element in convex set K by inductive hypothesis, and $\frac{1}{\sum_{j=1}^{N} \beta_j}$ is an element of the field $\mathbb{R}$, we have $\sum_{j=1}^{N+1} \beta_j x_j$ is indeed an element of the convex set K.

■

Theorem 2.14. *Let X be a linear space over the reals.*

(i) The empty set is convex.

Proof: *This is vacuously true.*

In any case, suppose otherwise. Suppose that $\emptyset$ is not convex. Then there are elements x and y in $\emptyset$ such that $\alpha x + (1-\alpha)y$ is not in $\emptyset$. However, this is a contradiction, since there are no elements in $\emptyset$. Hence, the empty set is convex.

■

(ii) A subset consisting of a single point is convex.

Proof: *Suppose x is the only element in some subset K of the linear space X. Notice that $x = x(\alpha + 1 - \alpha) = \alpha x + (1-\alpha)x$. Hence, a subset consisting of a single point is convex.*

■

(iii) Every linear subspace of X is convex.

Proof: *Pick an arbitrary linear subspace of X, say K. Then by the definition of a linear subspace, for any two arbitrary elements x and y in K, we have that $\alpha x + (1-\alpha)y$ is also an element of K. Since K was an arbitrary linear subspace of X, we have that every linear subspace of X is convex.*

■

(iv) The sum of two convex subsets is convex.

Proof: *A convex combination of two elements, z_1 and z_2 in the set $K_1 + K_2$ is $z = \alpha z_1 + (1-\alpha)z_1$, for $\alpha \in [0,1]$ We want to show that z is also an element of the set $K_1 + K_2$.*

Now,

$$\begin{aligned} z = \alpha z_1 + (1-\alpha)z_1 &= \alpha(x_1 + x_2) + (1-\alpha)(y_1 + y_2) \quad \text{for } x_1, y_1 \in K_1 \text{ and } x_2, y_2 \in K_2 \\ &= \Big(\alpha x_1 + (1-\alpha)y_1\Big) + \Big(\alpha x_2 + (1-\alpha)y_2\Big) \end{aligned}$$

Since K_1 is a convex subset, for elements x_1, y_1 in the set K_1, $\alpha x_1 + (1-\alpha)y_1$ is also an element of K_1. Similarly, since K_2 is a convex subset, for elements x_2, y_2 in the set K_2 we have $\alpha x_2 + (1-\alpha)y_2$ is also an element of K_2.

Hence, $z = \alpha z_1 + (1-\alpha)z_1$ is also an element of the set $K_1 + K_2$.

■

(v) *If K is convex, so is $-K$.*

Proof: *An arbitrary convex combination of elements from $-K$, is $z = \alpha(-x) + (1-\alpha)(-y)$ for elements $x, y \in K$. We want to show that z is also an element of $-K$*

But this is clear, since $z = -\Big(\alpha(x) + (1-\alpha)(y)\Big)$, and $\alpha x + (1-\alpha)y$ is an element of a convex set K.

■

(vi) *The intersection of an arbitrary collection of convex sets is convex.*

Proof: *Suppose that $\{K_\theta : \theta \in I\}$ is any collection of convex subsets of linear space X. Then a convex combination of elements from $\cap_{\theta \in I} K_\theta$ is $z = \alpha x + (1-\alpha)y$, for elements x, y in $\cap_{\theta \in I} K_\theta$. We want to show that z is also an element of $\cap_{\theta \in I} K_\theta$.*

But this is clear, since x, y is an element of a particular K_{θ_0}. Since, K_{θ_0} is a convex subset, we have that $\alpha x + (1-\alpha)y$ is also an arbitrary element of K_{θ_0}. But θ_0 was arbitrary. Hence the previous much be true for any $\theta_0 \in I$. Hence, $z = \alpha x + (1-\alpha)y$ is also an element of $\cap_{\theta \in I} K_\theta$.

■

(vii) *Let K_j be a collection of convex subsets that is totally ordered by inclusion. Then their union $\bigcup K_j$ is convex.*

Proof: *Suppose that $\{K_\theta : \theta \in I\}$ a totally ordered collection of convex subsets of linear space X. Then a convex combination of elements of $\bigcup_{\theta \in I} K_\theta$ is $z = \alpha x_1 + (1-\alpha)x_2$ for x_1 in K_{θ_1} and x_2 in K_{θ_2}. Without loss of generality, suppose that $K_{\theta_1} \subseteq K_{\theta_2}$. Then it is clear that x_1 in also an element of K_{θ_2}. Since K_{θ_2} is a convex subset, we have that $z = \alpha x_1 + (1-\alpha)x_2$ is also an element of K_2. Hence, z is an element of $\bigcup_{\theta \in I} K_\theta$.*

■

(viii) *The image of a convex set under a linear map is convex.*

Proof: *Fix a linear map T, such that $T : X \mapsto X^*$, where X and X^* are linear spaces.*

Now, a convex combination of elements of X^* is $z = \alpha z_1 + (1-\alpha)z_2$ from elements z_1 and z_2 in X^*. Then $z = \alpha T(x) + (1-\alpha)T(y)$ for elements x, y in X. Then by Definition 2.7, we have $\alpha T(x) + (1-\alpha)T(y) = T\Big(x + (1-\alpha)y\Big)$. Since $x + (1-\alpha)y$ is in the convex set X, we have that z is an element of X^*.

■

(ix) *The inverse image of a convex set under a linear map is convex.*

Proof: *Since the inverse image of a convex set under a linear space by Claim 2.10, we can apply (iii) to show that indeed, the inverse image of a convex set under a linear map also a convex set.*

■

Property 2.15 (Convex Sets)**.** *For a linear space X over a field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$*

1. *If Y is a linear subspace of X, then Y is convex.*

   Proof: *Pick arbitrary x_1 and x_2 in Y, α in $[0,1]$. Since Y is linear, $\alpha x_1 + (1-\alpha)x_2$ is in Y. Hence Y is convex.*

   ■

2. *For two* convex *subsets, Y_1 and Y_2 of X, $Y_1 + Y_2$ is also convex.*

   Proof: *Pick arbitrary elements z_1 and z_2 of $Y_1 + Y_2$, and α between $[0,1]$. Notice that $z_1 = x_1 + y_1$, and $z_2 = x_2 + y_2$, where x_1, x_2 in Y_1 and y_1, y_2 in Y_2.*

   Consider $\alpha z_1 + (1-\alpha)z_2 = \alpha(x_1+y_1) + (1-\alpha)(x_2+y_2) = \alpha x_1 + (1-\alpha)(x_2) + \alpha y_1 + (1-\alpha)(y_2)$.

   Notice that $\alpha x_1 + (1-\alpha)(x_2)$ is in Y_1 and $\alpha y_1 + (1-\alpha)(y_2)$ is in Y_2, since Y_1 and Y_2 are convex sets.

   In particular, $\alpha z_1 + (1-\alpha)z_2 = \alpha(x_1+y_1) + (1-\alpha)(x_2+y_2)$ is in $Y_1 + Y_2$. Hence, $Y_1 + Y_2$ is convex.

   ■

3. *For a family of* convex *sets of X, say K_θ for $\theta \in I$. Put $K = \bigcap_{\theta\in I} K_\theta$, then K is also convex.*

   Proof: *Pick x and y from K and α in $[0,1]$ and fix θ in I. Notice x and y are also in K_θ. Since K_θ is convex, we have $\alpha x + (1-\alpha)y$ is in K_θ.*

   Since θ was arbitrary, the previous must be true for any θ in I. That is, $\alpha x + (1-\alpha)y$ is in K.

   Hence, K is convex.

   ■

4. *Consider a family of **totally ordered** convex sets of X, say K_θ for $\theta \in I$. Recall Definition 1.10. Without loss of generality, assume that $K_{\theta_1} \subseteq K_{\theta_2}$. Put $K = \bigcup_{\theta\in I} K_\theta$, then K is also convex.*

   Proof: *Pick x and y from K and α in $[0,1]$, then there exists θ_1 and θ_2 such that x in is K_{θ_1} and y in is K_{θ_2}.*

   Since, $K_{\theta_1} \subseteq K_{\theta_2}$, we have that x is also in K_{θ_2}. Since K_{θ_2} is a convex set, we have, $\alpha x + (1-\alpha)y$ is in K_{θ_2}. In particular, since $K_{\theta_2} \subseteq K$, we have that $\alpha x + (1-\alpha)y$ is in K.

   Hence, K is convex.

   ■

5. *Fix a linear map, say $T : X \mapsto Y$. For a convex $K \subset X$, $T(K)$ is also convex.*

   Proof: *Pick z_1 and z_2 from K and α in $[0,1]$, then there exists x_1 and x_2 in K such that $T(x_1) = z_1$ and $T(x_2) = z_2$.*

   Since K is a convex set, we have that $\alpha x_1 + (1-\alpha)x_2$ is in K, hence $T(\alpha x_1 + (1-\alpha)x_2)$ is in $T(K)$.

   Since T is a linear map, we have $T(\alpha x_1 + (1-\alpha)x_2) = \alpha T(x_1) + T((1-\alpha)x_2)$. In particular $\alpha T(x_1) + T((1-\alpha)x_2) = \alpha z_1 + (1-\alpha)z_2$ is in $T(K)$.

   Hence, $T(K)$ is convex.

   ■

6. *Fix a linear map, say $T : X \mapsto Y$. For a convex $K \subset Y$, $T^{-1}(K)$ is also convex.*

   Proof: *Pick x_1 and x_2 from $T^{-1}(K)$ and α in $[0,1]$, then there exists $T(x_1)$ and $T(x_2)$ from K.*

   Since K is convex, we have $\alpha T(x_1) + T((1-\alpha)x_2)$ is in K.

   Since T is a linear map, we have that $\alpha T(x_1) + T((1-\alpha)x_2) = T(\alpha x_1 + (1-\alpha)x_2)$ is in K.

   In particular, $\alpha x_1 + (1-\alpha)x_2$ is in $T^{-1}(K)$.

   Hence, $T^{-1}(K)$ is convex.

   ■

2.3.1 Convex Hull

Definition 2.16. *For any subset S of linear space X over a field $\mathbb{K} = \mathbb{R}$, where S is* not necessarily *convex, consider a family of convex sets, $K_\theta \supseteq S$. We define that* ***convex hull*** *of a set S as $cu(S) = \bigcap_{\theta \in I} K_\theta$.*

Theorem 2.17.

(i) The convex hull of S is the smallest convex set containing S.

(ii) The convex hull of S consists of all convex combinations *of points of S.*

Claim 2.18. *$cu(S) = \bigcap_{\theta \in I} K_\theta$ is the* smallest *convex set which contains S. That is,*

1. *$cu(S)$ is convex,*
2. *$cu(S)$ contains S, and*
3. *If K is another set that contains S, then $K \supseteq cu(S)$*

Proof:

1. Recall by Property 2.15 that $cu(S) = \bigcap_{\theta \in I} K_\theta$ is a convex set.
2. Notice that each $S \subseteq K_\theta$. In particular $S \subseteq \bigcap_{\theta \in I} K_\theta$.
3. Suppose $S \subseteq K$, Since K is one of K_θ, we have that $cu(S) \subseteq K$. That is, $cu(S) = \bigcap_{\theta \in I} K_\theta$ is the smallest convex set which contains S.

■

Claim 2.19. $cu(S) = \left\{ \sum_{j=1}^{n} \alpha_j x_j : x_j \in S, \alpha_j \in [0,1], \sum_{j=1}^{n} \alpha_j = 1, n \textit{ is any natural number} \right\}$

Proof:

1. Put Z $= \left\{ \sum_{j=1}^{n} \alpha_j x_j : x_j \in S, \alpha_j \in [0,1], \sum_{j=1}^{n} \alpha_j = 1, n \text{ is any natural number} \right\}$.

   It is clear that $S \subseteq Z$, since if we take any element from S, say s, it can be written in the form $\sum_{j=1}^{1} \alpha s$, for $\alpha = 1$. In particular, s is also an element of Z.

   Pick z_1 and z_2 in Z and α in $[0,1]$. Notice that $z_1 = \sum_{j=1}^{N} \alpha_j x_j$ and $z_2 = \sum_{j=1}^{M} \beta_j y_j$.

   Now, $\alpha z_1 + (1-\alpha) z_2 = \alpha \sum_{j=1}^{N} \alpha_j x_j + (1-\alpha) \sum_{j=1}^{M} \beta_j y_j$.

   Notice, each x_j and y_j are elements of S. Furthermore, the sum, $\alpha \sum_{j=1}^{N} \alpha_j + (1-\alpha) \sum_{j=1}^{M} \beta_j = 1$, we have that $\alpha z_1 + (1-\alpha) z_2$ is a *convex combination* of elements of S. In particular, $\alpha z_1 + (1-\alpha) z_2 \in Z$. Therefore, Z is a convex set.

   Hence, Z is a convex set containing S, by Claim 2.18 Z also contains $cu(S)$.

2. Suppose K is another convex set containing S. Then from above, K must contain all convex combinations of elements of S. But, by the definition of Z, Z *only convex combinations* of elements of S. That is, K contains Z. Hence Z is the *smallest* convex set containing S.

By (1) and (2), we have that $cu(S) = Z$. That is, the convex hull of S consists of all *convex combinations* of points of S.

■

Definition 2.20 (Extreme Subset from [1])**.** *A subset E of a convex set K is called an* ***extreme subset*** *if:*

1. *E is convex and nonempty, and*
2. *whenever a point x of E is expressed as*

$$x = \frac{y+z}{2} \quad y, z \text{ in } K$$

then both y and z belong to E.

An extreme subset consisting of a single point is called an extreme point *of K.*

Example 2.21 (Example from [1])**.** *K is the interval $0 \leq x \leq 1$; the two endpoints are extreme points.*

Example 2.22 (Example from [1])**.** *K is the closed disk, $x^2 + y^2 \leq 1$. Every point on the circle $x^2 + y^2 = 1$ is an extreme point.*

Example 2.23 (Example from [1])**.** *The open disk, $x^2 + y^2 < 1$ has no extreme points.*

Example 2.24 (Example from [1])**.** *K a polyhedron, including faces. Its extreme subsets are its faces, edges, vertices, and of course K itself.*

Now, we introduce some useful theorems.

Theorem 2.25 (Theorem from [1])**.** *Let K be a convex set, E an extreme subset of K, and F an extreme subset of E. Then F is an extreme subset of K.*

Theorem 2.26 (Theorem from [1])**.** *Let M be a linear map of the linear space X into the linear space U. Let K be a convex subset of U, E an extreme subset of K. Then the inverse image of E is either empty or an extreme subset of the inverse image of K.*

By taking U to be one dimensional, we have the following corollary of Theorem 2.26:

Corollary 2.27 (Corollary from [1])**.** *Denote by H a convex subset of a linear space X, l a linear map of X into $\mathbb{R}$, H_{min} and H_{max} the subsets of H, where l achieves its minimum and maximum, respectively.*

Claim 2.28 (Claim from [1])**.** *When* nonempty*, H_{min} and H_{max} are extreme points.*

2.4 Problems from Section 2

Problem 2.29. *Prove Claim 2.6.*

Problem 2.30. *Show that each Example 2.12 is a convex set.*

Problem 2.31. *Prove Claim 2.13 by induction with the base case $N = 2$*

Problem 2.32. *Prove Theorem 2.14.*

Problem 2.33. *Prove Theorem 2.17.*

Problem 2.34. *Prove all claims from Examples 2.21 to 2.24.*

Problem 2.35. *Prove Theorem 2.25.*

Problem 2.36. *Prove Theorem 2.26.*

Problem 2.37. *Prove Corollary 2.27 as well as Claim 2.28.*

3 Normed Linear Spaces: Definition and Basic Properties

3.1 Normed Linear Space

Definition 3.1 (Normed Linear Space)**.** *Consider a linear space represented by X and a field $\mathbb{K}$ which will be either $\mathbb{R}$ or $\mathbb{C}$, we say that a* ***norm*** *represented by N is a function from $X \mapsto \mathbb{R}_+$ with the following properties:*

1. *$N(x)$ is* nonnegative *and $N(x) = 0 \iff x = 0$ for any x in X (definiteness),*
2. *for all α in K and for any x in X, $N(\alpha x) = |\alpha| N(x)$ (homogeneity),*
3. *for any two elements in X, say x and y, we have $N(x+y) \leq N(x) + N(y)$ (Subadditivity).*

A ***normed linear space*** *is a linear space equipped with a norm as defined above. For convenience, we represent $N(x)$ as $\|x\|$.*

Before we proceed, we have two claims:

Claim 3.2. *For x in X, we have $\|-x\| = \|x\|$*

Proof: This is clear, since $\|-x\| = |-1| \|x\| = \|x\|$.

∎

Remark 3.3. *It follow from Definition 3.1 for elements x, y, z in a normed linear space X that,*

$$\left\|(x+y)+z\right\| \leq \|x+y\| + \|z\| \leq \|x\| + \|y\| + \|z\| .$$

Furthermore, by induction, we have,

$$\left\|\sum_{i\in I} x_i\right\| \leq \sum_{i\in I} \|x_i\| \qquad \text{for } x_i \in X$$

Claim 3.4. *For two elements of X, say x and y, we have that $\Big|\|x\| - \|y\|\Big| \leq \|x-y\|$*

Proof: Notice that $\|x\| = \|x-y+y\|$. From the definition of a norm , we have that $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$. Therefore, $\|x\| - \|y\| \leq \|x-y\|$.

Furthermore, we have that $\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$ by Claim 3.2. The proof follows.

∎

Definition 3.5 (Distance)**.** *Consider a function $d : X \times X \mapsto \mathbb{R}_+$, we define the* ***distance*** *between two elements of X, say x and y as $d(x,y) = \|y-x\|$. By Definition 3.1 and Claims 3.2 and 3.4 the following properties hold:*

1. *$d(x,y) \geq 0$ and $d(x,y) = 0$* if *$y = x$.*
2. *$d(x,y) = d(y,x)$.*
3. *$d(x,z) \leq d(x,y) + d(y,z)$*

The following is a path of concepts which you will be familiar with:

($\|x\|$)→(distance)→(topology)→(sequences $x_n \to x \in \mathbb{R} \iff \|x_n - x\| \to 0$)→(open, closed and compact sets). These concepts will play an important role in the theory of linear spaces.

We can now discuss metrics which are equivalent.

Definition 3.6 (Equivalent Metrics)**.** *For a linear space X and two different norms, say $\|\cdot\|_1$ and $\|\cdot\|_2$, the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are* ***equivalent*** *if there exists $0 < c < \infty$ such that for any x in X, $c\|x\|_1 \leq \|x\|_2 \leq \frac{1}{c}\|x\|_1$.*

Claim 3.7. *For a linear space X and a given norm $\|\cdot\|$. If $Y \subseteq X$ linear subspace, then Y equipped with $\|\cdot\|$ is a* new *normed linear space.*

Proof: This is clear, since Y is a linear subspace of X. Hence, by Definition 3.1 we have that Y is a new normed linear space.

■

Claim 3.8 (Product Space)**.** *For linear spaces X_1 equipped with $\|\cdot\|_1$ and X_2 equipped with $\|\cdot\|_2$, we definition a new space, $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$. Notice that $X_1 \times X_2$ is a linear space. Scalar multiplication is defined by $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$. The following are three* possible *ways to define a norm on $X_1 \times X_2$:*

(i) $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$.

(ii) $\|(x_1, x_2)\| = \max\{\|x_1\|_1, \|x_2\|_2\}$.

(iii) $\|(x_1, x_2)\| = \sqrt{\|x_1\|_1^2 + \|x_2\|_2^2}$.

Proof: Apply Definition 3.1 to each to show each is a norm on $X_1 \times X_2$. ■

We now recall some elementary definitions from *Real Analysis* in the context of norms:

Definition 3.9 (Sequence)**.** *A* ***sequence*** *of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$. That is, for $f : \mathbb{N} \mapsto \mathbb{R}$ denoted $x_n = f(n)$, we write the sequence as an ordered n-tuple $(x_1, x_2, x_3, \cdots)$ or more compactly $(x_n)_{n\in\mathbb{N}}$.*

Definition 3.10 (Convergence)**.** *For a linear space X equipped with a norm $\|\cdot\|$, the sequence $(x_n)_{n\geq 1}$* ***converges*** *to an element x in X,*

if for any $\epsilon > 0$, there is at least one integer N such that for any $n \geq N$ we have $\|x_n - x\| \leq \epsilon$.

We then write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$ or $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 3.11 (Supremum and Infimum)**.** *The* ***Supremum*** *of a sequence, denoted $(x_n)_{n\in\mathbb{N}}$ in a linear space X, is defined as the least upper bound if such a sequence is bounded above by a real number, say x. If $(x_n)_{n\in\mathbb{N}}$ is unbounded above, then we say that the supremum of the sequence equals ∞.*

That is, if $(x_n)_{n\in\mathbb{N}}$ is an increasing *sequence in $\mathbb{R}$, we have,*

$$\sup\{x_n : n \in \mathbb{N}\} = \lim_{n\to\infty} x_n = x$$
$$\iff$$
$$\forall\epsilon > 0, \exists x_j \in \{x_n\} \text{ such that } x_j > x - \epsilon$$

*Similarly, the **Infimum** of a sequence, is defined as the greatest lower bound if such a sequence is bounded below by a real number. If $(x_n)_{n\in\mathbb{N}}$ is unbounded below, then we say that the infimum of the sequence equals $-\infty$.*

That is, if $(x_n)_{n\in\mathbb{N}}$ is n decreasing *sequence in $\mathbb{R}$, we have,*

$$\inf\{x_n : n \in \mathbb{N}\} = \lim_{n\to\infty} x_n = x$$
$$\Longleftrightarrow$$
$$\forall \epsilon > 0, \exists x_j \in \{x_n\} \text{ such that } x_j < x + \epsilon$$

Proposition 3.12 (from [2])**.** *For a normed linear space X and elements x_n, y_n, x, y in X, α_n, α in $\mathbb{K}$, the following are met:*

(i) The limit point x in Definition 3.10 is uniquely determined.

(ii) If $x_n \to x$ as $n \to \infty$, then the sequence (x_n) is bounded, that is, there is a number $r \geq 0$ such that $\|x_n\| \leq r$ for any n.

(iii) If $x_n \to x$ as $n \to \infty$, then,

$$\|x_n\| \to \|x\| \qquad \text{as } n \to \infty$$

(iv) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then

$$x_n + y_n \to x + y \qquad \text{as } n \to \infty$$

(v) If $x_n \to x$ and $\alpha_n \to \alpha$ as $n \to \infty$, then

$$\alpha_n x_n \to \alpha x \qquad \text{as } n \to \infty$$

Proof:

(i) Suppose a sequence x_n to x and y. Then, $0 \leq \|x - y\| = \|x - x_n + x_n - y\| \leq \|x - x_n\| + \|x_n - y\| \leq 0$ as $n \to \infty$. Hence, $x = y$.

(ii) By Definition 3.10, we have $\|x_n - x\| = 0$ as $n \to \infty$. Put $M \geq 0$. Then it is clear that M is an upper bound for $\|x_n - x\|$. Now, $\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\| \leq M + \|x\|$. That is, $\|x_n\|$ is bounded above by $r = M + \|x\|$.

(iii) From Claim 3.4, we have $\Big|\|x_n\| - \|x\|\Big| \leq \|x_n - x\| \to 0$ as $n \to 0$.

(iv) Suppose $x_n \to x$ and $y_n \to y$ as $n \to \infty$. We have $0 \leq \Big\|(x_n - y_n) - (x - y)\Big\| = \Big\|(x_n - x) + (y - y_n)\Big\| \leq \Big\|x_n - x\Big\| + \Big\|y_n - y\Big\| = 0$. By Definition 3.10, the proof follows.

(v) For a sequence $x_n \to x$ and $\alpha_n \to \alpha$ as $n \to \infty$, we have

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\| \\ &= \Big\|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\Big\| \\ &= \Big\|(\alpha_n - \alpha)x_n\Big\| + |\alpha| \Big\|(x_n - x)\Big\| = 0 \qquad \text{as } n \to \infty\end{aligned}$$

■

Claim 3.13. *For a linear subspace Y of linear space X over a field $\mathbb{K}$ and a given norm $\|\cdot\|$, denote $\overline{Y}$ as the* closure *of Y. That is, $\overline{Y} = \{x \in X : \exists (x_n)_{n\geq 1} \in Y \text{ and } x_n \to x\} \cup Y$. Then,*

(i) $\overline{Y}$ is a linear subspace and

(ii) $\overline{Y}$ equipped with $\|\cdot\|$ is a new normed linear subspace.

Proof:

1. Notice that we can find two sequences x_n and y_n in Y such that x and y are in $\overline{Y}$. Since Y is a linear subspace, the sequence $\alpha x_n + y_n$ in also in Y. Then by the definition of $\overline{Y}$, there is $\alpha x + y$ in $\overline{Y}$ such that $(\alpha x_n + y_n) \to (\alpha x + y)$ for some $\alpha \in \mathbb{K}$. Hence $\overline{Y}$ is a linear subspace of X.

2. Notice for two elements in $\overline{Y}$ say x, y, we have $|x_n - x| \leq \frac{\epsilon}{2|\alpha|}$ and $|y_n - y| \leq \frac{\epsilon}{2}$ for any $\epsilon > 0$ and $\alpha \in \mathbb{K}$. For now, suppose $\alpha \neq 0$.

   Then, $\left\|\alpha(x_n - x) + (y_n - y)\right\| = \left\|(\alpha x_n + y_n) - (\alpha x + y)\right\| \leq |\alpha| \left\|(x_n - x)\right\| + \left\|y_n - y\right\| \leq \frac{\epsilon}{2|\alpha|} + \frac{\epsilon}{2} = \epsilon.$

   The case when $\alpha = 0$, is clear, since $\left\|\alpha(x_n - x) + (y_n - y)\right\| = \left\|y_n - y\right\| \leq \epsilon$

■

Example 3.14. *Suppose Y is a linear subspace of X equipped with a norm $\|\cdot\|$, and Y is* closed*. Recall from a previous lecture, that we defined $X\mid_Y = \{[x] : x \in X\}$, where $[x] = \{z \in X : (z - x) \in Y\}$. Recall that $X\mid_Y$ is a linear space where addition and multiplication by scalars is defined.*

Define $\left\|[x]\right\| = \inf\{\|z\| : z \in [x]\}$. Then $\left\|[x]\right\|$ is a norm in $X\mid_Y$.

Proof: Two show that $\|[x]\|$ is a norm in $X\mid_Y$, we will show that it satisfies the three properties from Definition 3.1.

Before we begin, notice that for $[0] \in X\mid_Y$, we have $[0] = \{z \in X : (z - 0) \in Y\} = Y$.

1. $\|\cdot\| \geq 0$, is clear, since we are taking the infimum over nonnegative numbers. We want to show $\|[\cdot]\| = 0 \iff [\cdot] = [0]$.

   Pick the element, say $[x]$, from the quotient space $X\mid_Y$, such that $\|[x]\| = 0$. By our definition we have, $\inf\{\|z\| : z - x \in Y \text{ and } x, z \in X\} = 0$. Notice, $\|z_j\| \to 0$ implies that $z_j \to 0$, since for any $\epsilon > 0$ we have $\left|\|z_j\| - 0\right| = \|z_j\| = \|z_j - 0\| \leq \epsilon$ for $n \geq k_0$.

   Now, $z_j \to 0$ implies that the sequence $(z_j - x) \to -x$. Since Y is closed and $(z_j - x) \to -x$, we have that $-x \in Y$. Furthermore, since Y is a linear space, we have that x is also an element of Y.

   Claim 3.15. *If $x \in Y$ then $[x] = [0]$.*

   Proof: We will show that if $x \in Y$ then (i) $[x] \subseteq [0]$ and (ii) $[0] \subseteq [x]$.

   (i) Pick $z \in [x]$, then $z - x \in Y$. Notice that Y is a linear subspace, $x \in Y$ from above, and $z - x \in Y$, we have $z = z - x + x$ is also in an element of Y. Since $z \in Y$, we have that $z \in [0] = \{z : z = (z - 0) \in Y\}$. Hence, $[x] \subseteq [0]$.

(ii) Pick $z \in [0]$. Then $(z - 0) = z \in Y$. Since Y is a linear space and $z \in Y$, we have $z - x$ is also in Y. Hence, $z \in [x]$. That is, $[0] \subseteq [x]$.
By (i) and (ii), we have if $x \in Y$ then $[x] = [0]$.

Hence, we have that

$$\|[\cdot]\| = 0 \iff [\cdot] = [0] \tag{14}$$

■

2. For a scalar, $\alpha \in \mathbb{K}$, we want to show that $\Big\|\alpha \cdot [x]\Big\| = |\alpha| \cdot \Big\|[x]\Big\|$.

(i) From Claim 2.6, we have that $\alpha[x] = [\alpha x]$.
Notice, $\Big\|[\alpha x]\Big\| = \inf\{\|z\| : z \in [\alpha x]\}$ by the definition of $\Big\|[x]\Big\|$. But, $\inf\{\|z\| : z \in [\alpha x]\} = \inf\{\|z\| : z \in \alpha[x]\}$. So we have $\Big\|[\alpha x]\Big\| = \inf\{\|z\| : z \in \alpha[x]\}$.

(ii) Assume for now $\alpha \neq 0$. Then,

$$\Big\|[\alpha x]\Big\| = \inf\{\|z\| : z \in \alpha[x]\} = \inf\left\{\|z\| : \frac{z}{\alpha} \in [x]\right\}$$

But,

$$\begin{aligned}\inf\{\|z\| : z \in \alpha[x]\} &= \inf\left\{\|z\| : \frac{z}{\alpha} \in [x]\right\} \\ &= \inf\left\{|\alpha| \cdot \left\|\frac{1}{\alpha}z\right\| : \frac{1}{\alpha}z \in [x]\right\} \\ &= |\alpha| \inf\left\{\left\|\frac{1}{\alpha}z\right\| : \frac{1}{\alpha}z \in [x]\right\}\end{aligned}$$

Put $\frac{1}{\alpha}z = y$, then $|\alpha| \inf\{\|y\| : y \in [x]\} = |\alpha| \Big\|[x]\Big\|$ by definition. That is, $\Big\|\alpha \cdot [x]\Big\| = |\alpha| \cdot \Big\|[x]\Big\|$, when $\alpha \neq 0$.

(iii) Consider $\alpha = 0$. Then $|\alpha| \cdot \Big\|[x]\Big\| = 0$, since $|0| = 0$. While $\Big\|\alpha \cdot [x]\Big\| = \Big\|[\alpha x]\Big\| = \Big\|[0]\Big\| = 0$ by Eq. (14).

Hence by (i), (ii), (iii) we have,

$$\Big\|\alpha \cdot [x]\Big\| = \Big\|[\alpha x]\Big\| \quad \text{for any } \alpha \in \mathbb{K} \tag{15}$$

3. (i) Now, we want to show that $\Big\|[x] + [y]\Big\| \leq \Big\|[x]\Big\| + \Big\|[y]\Big\|$.
By definition, $\Big\|[x] + [y]\Big\| = \Big\|[x + y]\Big\| = \inf\{\|z\| : z \in [x + y]\}$. Then, $\inf\{\|z\| : z \in [x + y]\} = \inf\{\|z_1 + z_2\| : z_1 \in [x], z_2 \in [y]\}$. By the definition of the norm, we have $\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$.
Hence, $\|z_1 + z_2\| \leq \inf\{\|z_1\| + \|z_2\|\}$.

(ii) Now, we will show that $\inf\{\|z_1\| + \|z_2\|\} \leq \inf\{\|z_1\|\} + \inf\{\|z_2\|\}$.
If either $\inf\{\|z_1\|\}$ or $\inf\{\|z_2\|\}$ are ∞, then it is clear $\inf\{\|z_1\| + \|z_2\|\} \leq \infty$.

(iii) Suppose $\inf\{\|z_1\|\}$ and $\inf\{\|z_2\|\}$ are finite, fix $\epsilon > 0$, then we can find $w_1 \in [x]$ and $w_2 \in [y]$ such that
$\inf\{\|z_1\| : z_1 \in [x]\} \leq \|w_1\| \leq \inf\{\|z_1\| : z_1 \in [x]\} + \frac{\epsilon}{2}$ and,
$\inf\{\|z_2\| : z_2 \in [y]\} \leq \|w_2\| \leq \inf\{\|z_2\| : z_2 \in [y]\} + \frac{\epsilon}{2}$
Then,

$$\inf\{\|z_1 + z_2\| : z_1 + z_2\} \leq \|w_1\| + \|w_2\| \leq \inf\{\|z_1\| : z_1 \in [x]\} + \inf\{\|z_2\| : z_2 \in [y]\} + \epsilon$$

Hence, $\inf\{\|z_1\| + \|z_2\|\} \leq \inf\{\|z_1\|\} + \inf\{\|z_2\|\}$.

Hence by (1), (2), (3), $\|[x]\| = \inf\{\|z\| : z \in [x]\}$ is a norm in $X\,|_Y$.

■

3.2 Banach Space

We begin with some preliminaries put into the context of linear spaces equipped with a norm that you should be familiar with. For convenience, we now write the sequence $(x_n)_{n\geq 1} = (x_1, x_2, \cdots, x_j, \cdots)$ as simply (x_n).

Definition 3.16 (Cauchy Sequence)**.** *For a linear space X equipped with a norm $\|\cdot\|$, the sequence (x_n) is a sequence is a* ***Cauchy Sequence****,*

if for any $\epsilon > 0$, there is at least one integer k_0, such that $\|x_j - x_k\| \leq \epsilon$ for any $k, j \geq k_0$.

Remark 3.17 (Relationship between Convergence and Cauchy Sequences)**.** *Cauchy sequences are intimately related to convergent sequences. For example, every convergent sequence in a normed linear space X is also a Cauchy sequence, since if $x_n \to x$, then*

$$\|x_n - x_m\| = \|x_n - x + x - x_m\| \leq \|x_n - x\| + \|x - x_m\| < 2 \times \frac{\epsilon}{2} = \epsilon.$$

The previous theorem is an elementary result in real analysis that will be useful for completing normed linear spaces.

Theorem 3.18. *Every Cauchy sequence is bounded.*

Proof: Recall, $\Big|\|x\| - \|y\|\Big| \leq \|x - y\|$. Then $\|x_n\| - \|x_m\| \leq \|x_n - x_m\| \leq \epsilon$ for any $\epsilon > 0$ and any $n, m \geq k_0$.

Now, fix $m = k_0$. Then, $\|x_n\| \leq \epsilon + \|x_{k_0}\|$ for any $n > k_0$. Put $M = max\{\|x_1\|, \|x_2\|, \cdots, \|x_{k_0}\| + \epsilon\}$. Hence, $\|x_n\| \leq M$. That is every Cauchy sequence is bounded.

■

Definition 3.19 (Complete Space)**.** *X is a* ***complete space*** *if* every *Cauchy sequence converges in X.*

Notice that a Cauchy sequence may not converge in a given space. E.g., take sequence $x_n = \frac{1}{n}$ in the interval $(0, 1)$. Then it is clear that x_m is Cauchy, but x_n converges to 0 which is *not* in the given space.

Proposition 3.20. *In a normed space, each convergent sequence is Cauchy.*

Proof: For a convergent sequence x_n in a normed linear space X, we have $\|x_n - x\| \leq \frac{\epsilon}{2}$ for some $n \geq k_0$.

Then, $\|x_n - x_m\| = \Big\|(x_n - x) + (x - x_m)\Big\| \leq \Big\|x_n - x\Big\| + \Big\|x - x_m\Big\| \leq \epsilon$ for $n, m \geq k_0$. ■

Remark 3.21 (Complete Normed Linear Space)**.** *Because Cauchy sequences are the sequences whose terms grow close together, the fields where all Cauchy sequences converge are the fields that are not "missing" any numbers. Furthermore, any divergent sequence is "truly" divergent, that is here is no bigger normed linear space which makes it convergent.*

In the case of the real line, every Cauchy sequence converges; *that is, being a Cauchy sequence is sufficient to guarantee the existence of a limit on the real line.* In the general case, however, this is not so *[1].*

Definition 3.22 (Banach Space)**.** *A* ***Banach Space*** *is a* normed linear space *which is also* complete*. That is,* *If (x_n) is a Cauchy sequence in a normed linear space X, then there is at least one x in X such that $x_n \to x$.*

Banach spaces are also called complete normed spaces [2].

Remark 3.23. *From Proposition 3.20, we get the following so-called Cauchy convergence criterion:* In a Banach space, a sequence is convergent if and only it is Cauchy.

Example 3.24 (Example from [1])**.** *Show that if X is Banach space, Y a closed subspace of X, the quotient space $X\,|_Y$ is complete.*

Hint 1. *Use a Cauchy sequence (q_n) in $X\,|_Y$ that satisfies $\|q_n - q_{n+1}\| < \frac{1}{n^2}$.*

How do I use this hint?!

3.3 Examples of Normed Linear Spaces

We now describe a number of the most important normed linear spaces from [1].

Example 3.25 (Example from [1])**.** *The space of all vectors with infinite number of components, $x = (a_1, a_2, \cdots) \in X$ where a_j is complex and $|a_j|$ are bounded. The norm is*

$$|x|_\infty = \sup_j |a_j| \tag{16}$$

This space is denoted as l^∞; it is complete.

Example 3.26 (Example from [1])**.** *The space of all vectors with infinitely many components such that $\sum |a_j|^p < \infty$, p some fixed number ≥ 1. The norm is*

$$|x|_p = \left(\sum |a_j|^p\right)^{\frac{1}{p}} \tag{17}$$

This space is denoted l^p; it is complete.

Example 3.27 (Example from [1])**.** *S an abstract set, X the space of all complex-valued functions f that are bounded. The norm is*

$$|f|_\infty = \sup_S |f(s)| \tag{18}$$

This space is complete.

Example 3.28 (Example from [1])**.** *Q a topological space, X the space of all complex valued, continuous, bounded functions f on Q. The norm is*

$$|f| = \sup_Q |f(q)| \tag{19}$$

This space is complete.

Example 3.29 (Example from [1])**.** *Q a topological space, X the space of all complex valued, continuous functions f with compact support. The norm is*

$$|f|_{max} = \max_Q |f(q)| \tag{20}$$

This space is not complete *unless Q is compact.*

Example 3.30 (Example from [1]). *D some domain in $\mathbb{R}^n$, the the space of all C^∞ functions f in D with the following property: for some integer k and $p \geq 1$,*

$$\int_D |\partial^\alpha f|^p \, dx < \infty \quad \text{for all } |\alpha| \leq k,$$

Where ∂^α is any partial derivative:

$$\partial^\alpha = \partial_1^{\alpha^1} \cdots \partial_n^{\alpha^n}, \quad \partial_j = \frac{\partial}{\partial x^j}, \quad |\alpha| = \alpha^1 + \cdots + \alpha^n$$

The norm is

$$|f|_{k,p} = \left(\sum_{|\alpha| \leq k} \int |\partial^\alpha f|^p \, dx \right)^{\frac{1}{p}} \tag{21}$$

Theorem 3.31 (from [1]). *The norms defined in Examples 3.25 to 3.30 have properties from Definition 3.1 imposed on a norm.*

3.4 Problems from Section 3

Problem 3.32. *Show that the properties enumerated in Definition 3.5 hold.*

Problem 3.33. *Prove Claim 3.7*

Problem 3.34. *Show that each possible definition of a norm of a product space from Claim 3.8 has the properties of a norm and that all three are equivalent by Claim 3.7.*

Problem 3.35. *Prove Claim 3.13.*

Problem 3.36. *Prove Example 3.24.*

4 Completing a Normed Linear Space (NLS)

Before we consider the process of completing a normed linear space, we first consider some normed linear space that are *not* complete.

Example 4.1. *Consider the space of continuous functions on the closed interval $[a, b]$ denoted $C[a, b]$. On this space, introduce a norm, say the L^1 norm. That is, $\|f, g\| = \int_a^b |f(x) - g(x)| \, dx$ for $f, g \in C[a, b]$. This normed linear space is* not *complete.*

Hint 2. *For a sequence not to be Cauchy, there needs to be some $N > 0$ such that for some $\epsilon > 0$, there are $m, n > N$ where $\|a_n - a_m\| > \epsilon$. In other words, no matter how far out into the sequence the terms are, there is no guarantee they will be close together.*

Solution (Example 4.1). *It is clear that this space is* not *complete. Indeed, we can find a sequence in such a space such that this sequence will not converge.*

For instance, put $[a, b] = [0, 1]$ and $f_n(x) = nx$. Then,

$$\|f_n, f_m\| = \int_0^1 |f_n(x) - f_m(x)| \, dx = \int_0^1 |nx - mx| \, dx = \left[|m - n| \frac{x^2}{2} \right]_0^1 = \frac{|m - n|}{2}$$

For $m = n + 1$, we always have $\|f_n, f_m\| = \frac{1}{2}$. That is by the hint provided in Hint 2, we have some $N > 0$ such that for $\epsilon = \frac{1}{4}$, there are $m, n > N$ we have $\|f, g\| > \epsilon$.

Example 4.2. *Let $X = (0, 1)$ and take the norm between any two numbers x and y belonging to X to be $\|x, y\| = |x - y|$. This normed linear space is not complete.*

Solution (Example 4.2)**.** *Take $x_n = \frac{1}{n}$. Notice that x_n is a Cauchy sequence that converges to 0. However, also notice that 0 is* not *in X. That is X is* not *a complete normed linear space.*

Notice that convergence is defined in terms of a limit. For example, take a convergent sequence $x_n \to x$ in a normed linear space X, and remove the point x from X, denoted $X \setminus x$. That is, assume for any n, $x_n \neq x$. Then (x_n) is still a sequence in the subspace $X \setminus x$, but it no longer converges. So (x_n) converges in X but *not* in $X \setminus x$. How are we to know whether a normed linear space has the property that *all* sequences are Cauchy sequences? If a normed linear space is "missing" these limit points, is it possible to add them into the normed linear space X?

The processes of adding these "missing" points is called *Completion of a Normed Linear Space.*

4.1 The Process of Completion of a Normed Linear Space

This lecture presents the standard method of completing a normed linear space. The construction does not differ from the one employed to complete a metric space found in [3] and is inspired from the construction of the real numbers by Cantor.

We would like to define a method to complete a *normed linear space*, that is we will introduce new points in the space to make it complete. This is based on a general method as follows:

Step 1 Consider a *Normed Linear Space (NLS)* represented by X such that X is *not necessarily* complete with respect to a given norm.

Step 2 Define a new space, Z, of Cauchy sequences from X, that is

$$Z = \{(x_j)_{j\geq 1} : x_j \in X \text{ and } (x_j)_{j\geq 1} \text{ is a Cauchy sequence}\}$$

Notice that Z is a linear subspace of X, since we can add two sequence in Z by adding each coordinate, and if each sequence is Cauchy, then the sum is also Cauchy. Hence Z is closed under addition. For the same reasons, Z is also closed under scalar multiplication.

Remark 4.3. *If we stopped here, we may be tempted to define a norm as follows:*

For a Cauchy Sequence $(x_j)_{j\geq 1}$ in the previously defined linear space Z, define $\|(x_j)_{j\geq 1}\| = \lim_{n\to\infty} \|x_j\|$.

Since $(x_j)_{j\geq 1}$ is a Cauchy sequence, we know that $(x_j)_{j\geq 1}$ will converge.

Notice that $\|(x_j)\| = \lim_{j\to\infty} \|x_j\| = 0$ does not *imply that $(x_j)_{j\geq 1} = (0)_{j\geq 1}$ it* only *implies $x_j \to 0$.*

Hence, $\|(x_j)\| = \lim_{n\to\infty} \|x_j\|$ is **not** *a norm on Z as defined in Definition 3.1.*

Step 3 Now, define a subspace of Z, denoted X_0, as the space of all *constant* sequences. That is, $X_0 = \{(x, x, x, \cdots) : x \in X\}$. It is clear that each constant sequence in X_0 is Cauchy. Furthermore, there is a one-to-one correspondence between X and X_0. In this way, we have *"embeded"* X into Z. That is, $X_0 \subseteq Z$, where X_0 represents the original set X.

Step 4 We wish to define a norm on Z. To define such a norm, we will need to introduce equivalence relations on Z.

We will say that two sequence, $(x_j)_{j\geq 1}$ and $(y_j)_{j\geq 1}$ are equivalent, denoted $(x_j)_{j\geq 1} \sim (y_j)_{j\geq 1}$, if $y_j - x_j \to 0 \iff \lim_{j\to\infty}(y_j - x_j) = 0$.

That is, we can create a new *linear subspace* $Y \subseteq Z$ such that $Y = \{(x_j)_{j\geq 1} \in Z : x_j \to 0\}$. Then $(x_j)_{j\geq 1}$ and $(y_j)_{j\geq 1}$ are equivalent if we have $(y_j)_{j\geq 1} - (x_j)_{j\geq 1} = (y_j - x_j)_{j\geq 1}$ is in Y. Notice that Y is a linear subspace of Z since it is closed under addition and scalar multiplication.

Step 5 Define $\overline{X} = Z\,|_Y = \{[(x_j)_{j\geq 1}] : (x_j)_{j\geq 1} \in Z\}$. Notice that elements of $\overline{X}$ are equivalence classes. That is $[(x_j)_{j\geq 1}] = \{(z_j)_{j\geq 1} : z_j - x_j \to 0\}$. We will show that the *closure* of X_0, denoted $\overline{X_0}$, is exactly $\overline{X} = Z\,|_Y$.

Step 6 Define $\left\|[(x_j)_{j\geq 1}]\right\| = \lim_{j\to\infty} \|x_j\|$. For convenience, we now write the sequence $(x_j)_{j\geq 1}$ as simply (x_j).

Claim 4.4.

(i) $\lim_{j\to\infty} \|x_j\|$ *is well defined.*

(ii) $\left\|[(x_j)]\right\|$ *is a norm as defined in Definition 3.1.*

Proof:

(i) For each Cauchy Sequence (x_j), we have that the sequence of norms $(\|x_j\|)$ is also Cauchy, since, for any $\epsilon > 0$, there is some k_0 such that $\left|\|x_j\| - \|x_k\|\right| \leq \|x_j - x_k\| \leq \epsilon$ for any $k, j \geq k_0$ by Claim 3.4. Furthermore, since the sequence of norms $(\|x_j\|)$ is a Cauchy sequence of real numbers, by Remark 3.21 we have that the sequence of norms, $(\|x_j\|)$, converges.

Now, take two sequences $(x_j)_{j\geq 1}$ and $(y_j)_{j\geq 1}$ both in $[(x_j)]$, such that $[(x_j)] = [(y_j)]$. We want to show that $\lim_{j\to\infty} \|x_j\| = \lim_{j\to\infty} \|y_j\|$.

Since we assumed that $[(x_j)] = [(y_j)]$, we have $[(y_j)] \sim [(x_j)] \iff y_j - x_j \to 0$. Then $\|y_j - x_j\| \to 0$. By Claim 3.4, we have $\left|\|y_j\| - \|x_j\|\right| \to 0$. Hence, $\lim_{j\to\infty} \|x_j\| = \lim_{j\to\infty} \|y_j\|$.

That is $\lim_{j\to\infty} \|x_j\|$ is well defined.

(ii) Now we will show that all three properties of a norm is satisfied.

(1) It is clear that $\left\|[x_j]\right\| \geq 0$ since each $\|x_j\| \geq 0$. Hence $\left\|[x_j]\right\| = \lim_{j\to\infty} \|x_j\| \geq 0$.

Now, assume that $\|[x_j]\| = 0$, then by definition we have $\|[x_j]\| = \lim_{j\to\infty} \|x_j\| = 0$.

We claim that the above means that $[x_j] = [0]$. That is, $(x_j) \in Y$.

Now, $\|x_j - 0\| = \|x_j\|$, then $\lim_{j\to\infty} \|x_j - 0\| = \lim_{j\to\infty} \|x_j\| = 0$, since $(x_j) \in Y$.

That is, $[x_j] = [0]$.

(2) Pick some scalar, say α. Now, $\left\|\alpha[x_j]\right\| = \left\|[\alpha x_j]\right\| = \lim_{j\to\infty} \|\alpha x_j\| = |\alpha| \lim_{j\to\infty} \|x_j\| = |\alpha| \left\|[x_j]\right\|$

Hence, $\left\|\alpha[x_j]\right\| = |\alpha| \left\|[x_j]\right\|$.

(3) Recall that $[x_j] + [y_j] = [x_j + y_j]$.

Then $\left\|[x_j]+[y_j]\right\| = \left\|[x_j+y_j]\right\| = \lim_{j\to\infty} \|x_j + y_j\| = \lim_{j\to\infty} (\|x_j + y_j\|) = \lim_{j\to\infty} \|x_j\| + \lim_{j\to\infty} \|y_j\| = \left\|[x_j]\right\| + \left\|[y_j]\right\|$.

Hence, $\left\|[x_j]\right\| = \lim_{j\to\infty} \|x_j\|$ is indeed a norm in the linear space $\overline{X}$.

■

Step 7 Finally, we will show that $\overline{X} = Z\,|_Y$ is complete with respect to the norm $\left\|[x_j]\right\| = \lim_{j\to\infty} \|x_j\|$.

Theorem 4.5. *$\overline{X}$ is complete with respect to the previously defined norm $\|\cdot\|$.*

Proof: Consider a sequence of equivalent classes of Cauchy sequences, $[x_j^n]$. We want to show there is at least one $[x_j]$ in $\overline{X} = Z\,|_Y$ such that $[x_j^n] \to [x_j]$ as $n \uparrow \infty$.

Aside: $[x_j^n]$ is a sequence of sequences, that is:

$$\left.\begin{array}{c}(x_j^1)_{j\geq 1} = (x_1^1, x_2^1, x_3^1, \cdots)\\ (x_j^2)_{j\geq 1} = (x_1^2, x_2^2, x_3^2, \cdots)\\ (x_j^3)_{j\geq 1} = (x_1^3, x_2^3, x_3^3, \cdots)\\ \vdots \\ (x_j^m)_{j\geq 1} = (x_1^m, x_2^m, x_3^m, \cdots)\\ \vdots\end{array}\right\} \text{each of these are Cauchy sequences.}$$

Then $[(x_j^n)]$ is Cauchy if for any $\epsilon > 0$ there is at least one M_0 such that $\left\|[(x_j^n)] - [(x_j^m)]\right\| \leq \epsilon$ for any $n, m \geq M_0$. That is, $\lim\limits_{j\to\infty} \left\|x_j^n - x_j^m\right\| \leq \epsilon$.

Returning to our proof, we want to show that $[x_j^n] \to [x_j]$. That is, $\lim\limits_{n\to\infty} \lim\limits_{j\to\infty} \left\|x_j^n - x_j\right\| = 0$.

(1) Pick $[x_j^n]$ Cauchy. So, for any $\epsilon > 0$ there is a N_0 such that $\left\|[x_j^n] - [x_j^m]\right\| = \lim\limits_{j\to\infty} \left\|x_j^n - x_j^m\right\| \leq \epsilon$ for every $n, m \geq N_0$.

Now, fix $p = 1, 2, 3\cdots$ and put $\epsilon = \frac{1}{2^{p+1}}$. Then there is $n(p)$ such that $\left\|[x^n] - [x^m]\right\| \leq \frac{1}{2^{p+1}}$ for $n, m \geq n(p)$.

Notice without loss of generality, we can assume that $n(p) < n(p + 1)$.

So, $\lim\limits_{j\to\infty} \left\|x_j^{n(p)} - x_j^m\right\| \leq \frac{1}{2^{p+1}}$.

Now, for $m = n(p + 1)$, there is an integer $R(n(p), m)$ such that, $\left\|x_j^{n(p)} - x_j^m\right\| \leq \frac{1}{2p}$ for $j \geq R(n(p), m)$.

■

4.2 Problems from Section 4

Problem 4.6. *Prove Theorem 3.31.*

Problem 4.7. *Show Example 4.1 is* not *complete.*

Problem 4.8. *Show Example 4.2 is not a complete.*

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